

**A NEW METHOD TO EVALUATE THE STRESS CONCENTRATION  
FACTOR OF A PLATE WITH DRILLED HOLES**



Bachelor's thesis

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ABSTRACT

The thesis introduces a new method of estimating the stress concentration factor in a plate with two or three drilled holes. It is known, that stress concentrations are one of the biggest problems to avoid when designing any kind of structure. One drilled hole is decreasing the strength of a whole structure at least three times, which makes the production cost and amount of needed material more, and to avoid stress concentration different methods are used, for example, adding more material (e.g. welding) or other connections (e.g. bolts, rivets, etc.), which creates more stress concentrations.

In this work a new approach is checked. It states, that more holes drilled near the main hole will decrease the stress concentration factor of the whole structure, which happens because of the stress interaction. But for now only few global recommendations have been given in previous studies, so this phenomena was studied more in cases with two and three drilled holes.

Body force method, which is another numerical method to solve engineering problems, was used to make a Matlab code, which can be easily used by anyone to evaluate the stress concentration factors. Finite element analysis confirmed the code working correctly and at the end it was confirmed that auxiliary holes can reduce stress concentration factor from value 3 up to value 2.3 at least. Few recommendations were given and new questions were set for further studies.

**Keywords** stress concentration, drilled hole, stress interaction, body force method

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## NOTATION

### Latin Symbols

$a, b$	Semi-minor and semi-major width of an ellipse
$E$	Modulus of elasticity
$G$	Shear modulus
$K_t$	Stress concentration factor
$MM$	Total number of segments
$N$	Numerical order of a segment
$r$	Radius of a hole or polar coordinate
$t$	Projected width of an ellipse
$u, v$	Displacement components in coordinate directions
$x, y$	Cartesian coordinates

### Greek Symbols

$\gamma$	Shear strain
$\varepsilon$	Normal strain
$\vartheta$	Polar coordinate
$\nu$	Poisson's ratio
$\rho$	Root radius
$\rho_{xN}, \rho_{yN}, \rho_X, \rho_Y$	Body force densities
$\sigma$	Normal stress or normal stress influence coefficients
$\tau$	Shear stress or shear stress influence coefficients
$\varphi$	Elliptical parameter

## 1 INTRODUCTION

Nowadays there are any type of parts for any type of purposes produced by engineers. Huge variety of materials and geometrical shapes are used. Engineers are coming with new methods all the time how to get the best advantages from the materials so the profit is maximized. Mostly it all leads to get the best possible strength-cost ratio, for example. Engineers face the number of challenges doing that.

One of the most significant tasks is to design needed part so that it will not break or deform too much. Here comes the importance of stress calculations. There are many factors, which must be taken into consideration: size of a part, dimensions of notches, radii, and holes, connections of several parts. Everything is affecting the stress value, which can the material stand before breaking or deforming plastically. All these non-homogeneities are creating such called stress concentration points, which are making the part weaker.

One important thing, which is creating these stress concentration points, is a hole. Depending on the position of the hole and the distance from hole to the edge of a part, the specimen itself can become up to about three times weaker or even more. To know how much stress is concentrated in that critical point, stress concentration factor  $K_t$  is used.

As there is often some hole needed in design purposes, engineers started to invent different ways how to minimize consequences of the stress concentration. One solution is adding the material – to use some welding, or other type of connecting the materials, or making the part itself larger. Everything of that leads to two major disadvantages – a) new critical points of the stress concentration are created and b) it makes production more costly due to increased work time and material consumption.

When technologies started to improve and it was possible to solve many difficult equations much faster using computers, new methods to calculate and estimate stress concentration appeared. Using so called body force method, which was invented in Japan in the second half of 20th century, it was discovered that stress concentration factor could be minimized by using auxiliary holes. It means that for reducing the effect of stress concentration better idea might be to remove material, not to add it.

There were some previous studies on that topic done and it is possible to find some general works with equations how to act for better results, but there were no detailed recommendations of the size, the position and the number of auxiliary holes for better and stronger design of mechanical parts, which might be very useful for companies to use. The idea of this

thesis is to generate some code, which will be easy to use for estimating the stress concentration factor as well as have concrete recommendations as it was said before.

### 1.1 Objective

The objective of this thesis is to come up with some recommendation of size and position of auxiliary holes so that stress concentration factor in a plate is minimized as much as possible.

## 2 BACKGROUND INFORMATION

### 2.1 General

Any part made from any material, for example, machine member, plate, etc. very often has regions where stress has significantly larger value than predicted in theory. This happens due to several reasons:

- Geometrical discontinuities or stress raisers such as holes, notches, and fillets;
- Internal microscopic non-homogeneities of the material created by some manufacturing processes such as casting, molding, etc.;
- Surface non-homogeneities such as cracks and marks created while producing part.

Therefore, definition of stress concentration is following: localized stress considerably higher than average (even in uniformly loaded cross sections of uniform thickness) due to abrupt changes in geometry or localized loading (ASME BTH-1/2005, 13).

Stress concentration is also called as “stress raiser” or “stress riser”.

Easy way to describe what actually a stress concentration means is if stress is considered as a flow. As we can see in figure 1, when there are no stress concentrations in a plate, stress flow is straight and it is not disturbed by anything. It means, if any point of the plate is examined, the stress value will always be the same and it will be equal to  $\sigma$ , which is external stress due to loading.

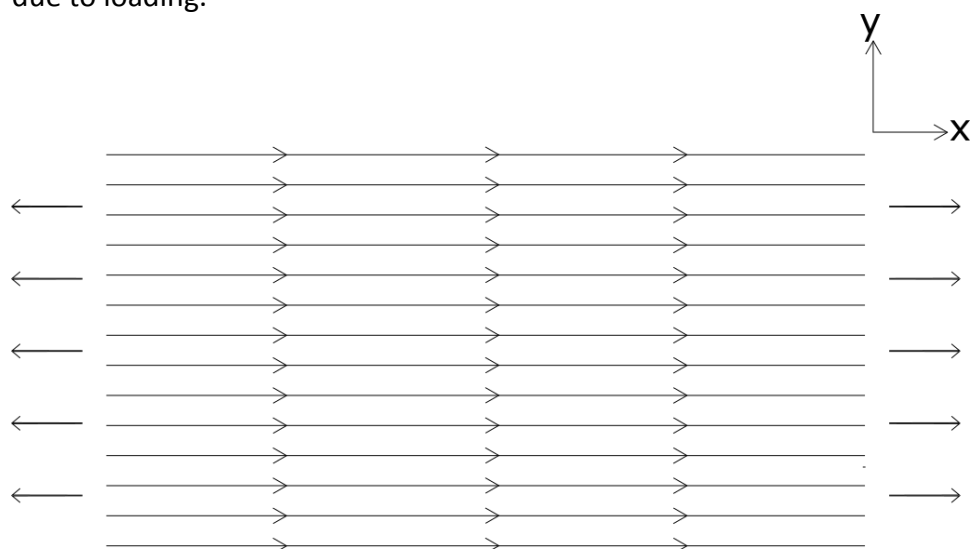


Figure 1. Stress flow in an infinite size plate

When there is one hole in the middle of the plate, stress flow needs somehow to avoid it, as stress cannot go outside of the plate. In these points, where stress flow is disturbed, stress concentration occurs. It

means that stress at that point will not be equal to the nominal value, to the external applied stress. Equation (1) is showing how stress concentration factor is defined.

$$K_t = \frac{\sigma_{MAX}}{\sigma_{NOM}} \quad (1)$$

$\sigma_{MAX}$  in the equation stands for maximum value of stress which occurs due critical points in the plate (in our case it is a hole) and  $\sigma_{NOM}$  – nominal value which is equal to external stress due loading and occurs in a plate without holes or in a plate with hole far enough from the stress concentration. As it is clear from above mentioned, stress concentration effect is local, so at some particular distance from the hole (depending on its size) there will be no stress raiser anymore.

From figure 2 it is easily seen what stress concentration means as stress flow differs from the one shown in figure 1.

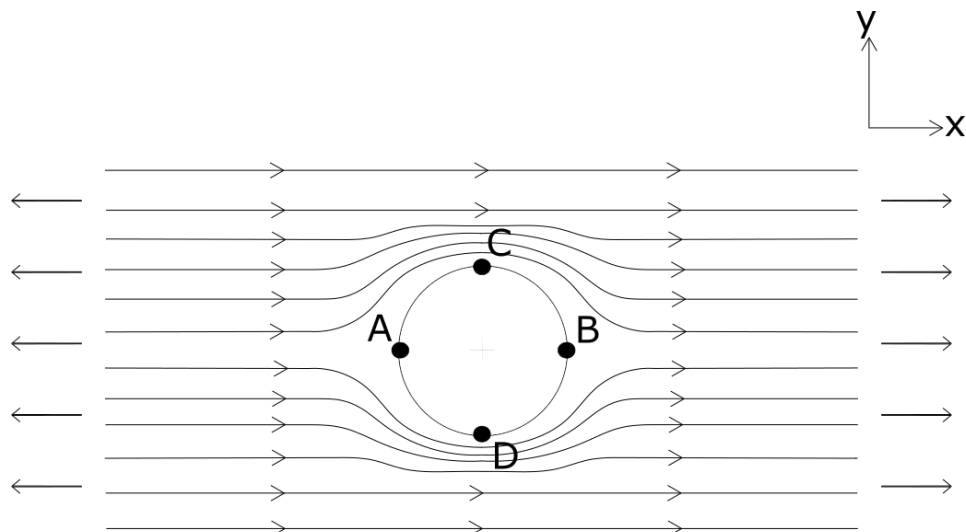


Figure 2. Stress flow in an infinite size plate with one drilled hole

Just looking on the picture, a prediction may be done, that critical stress concentration values will be at points A, B, C, and D, as disruption of the stress flow is the largest there. The conclusion can be done, that the bigger is the change of that flow, the larger stress concentration's value will be. Later, we will come back to this concept of stress concentration, when specimen for tests will be chosen.

## 2.2 Theory of Elasticity

As it is understandable from the name of the science, it works with the elastic region of the metals. Elasticity is concerned of the stresses and strains in a body due to applied loading, which can be mechanical or thermal, in those cases, when body reverts to its original shape and size

after releasing the loading. Comparing with mechanics or strength of materials, there are couple of benefits:

- Theory of Elasticity makes no physical assumption;
- Theory of Elasticity develops solution from:
  - Newton's laws of motion;
  - Euclidian geometry;
  - Material constitutive law (for example, Hooke's law);
- Superposition is allowed as well.

To find the stress concentration factor by means of theory elasticity, the superposition method is used (fig. 3).

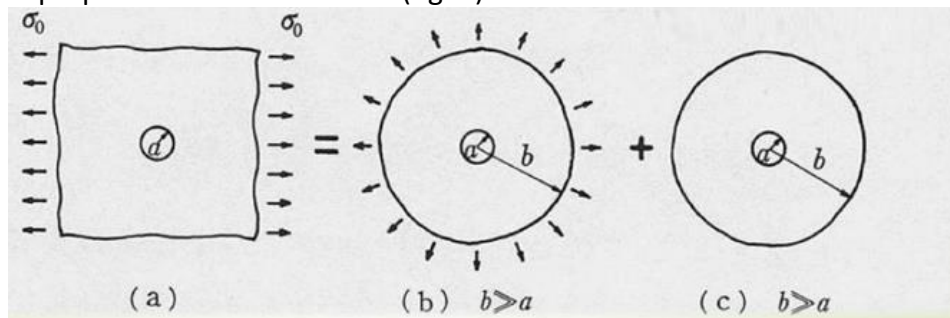


Figure 3. Stress concentration at a circular hole (Murakami n.d., 62)

As in this problem the plate is of an infinite size, we can “cut” any other shape so that the part will be still of an infinite size and there will be area, where no interaction of the stress flow will appear because of the hole. In this case, the round shape is chosen because in polar coordinate system solution can be found.

The stress values for cases (b) and (c) from figure 3 are the following

$$\text{CASE b) } \sigma_r = \frac{1}{2} \sigma_0 \quad (2a)$$

$$\text{CASE c) } \sigma_r = \frac{1}{2} \sigma_0 \cos 2\theta \quad (2b)$$

$$\text{CASE c) } \tau_{r\theta} = -\frac{1}{2} \sigma_0 \sin 2\theta \quad (2c)$$

where  $\sigma_0$  is the applied remote stress.

Ernst Gustav Kirsch, a German engineer, is primarily known for his equations, which describe the elastic stresses around the hole in an infinite plate in one directional tension.

The equation, which is needed in this problem, is following

$$\phi(r, \theta) = (Ar^2 + Br^4 + Cr^{-2} + D) \cos 2\theta \quad (3)$$

Hence, the stresses for our particular problem are following

$$\sigma_{rr} = -(2A + 6Cr^{-4} + 4Dr^{-2}) \cos 2\theta \quad (4a)$$

$$\sigma_{\theta\theta} = -(2A + 12Br^2 + 6Cr^{-4}) \cos 2\theta \quad (4b)$$

$$\tau_{r\theta} = -(2A + 6Br^2 - 6Cr^{-4} + 2Dr^{-2}) \sin 2\theta \quad (4c)$$

After some arithmetical manipulations, final equations for stresses are derived and are the following

$$\sigma_{rr} = \frac{\sigma_0}{2} \left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_0}{2} \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) \cos 2\theta \quad (5a)$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2} \left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_0}{2} \left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta \quad (5b)$$

$$\tau_{r\theta} = -\frac{\sigma_0}{2} \left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right) \sin 2\theta \quad (5c)$$

To find stress  $\sigma_{\theta\theta}$  in the x direction at “top” point of a hole (point C on figure 2), the  $\theta = \pi/2$  value is used, because in that case  $\sigma_x = \sigma_{\theta\theta}$  (fig. 4).

After simplifying the equation (5b), the following simple formula is got

$$\sigma_{\theta\theta} = \sigma_0 \left(1 + \frac{a^2}{2r^2} + \frac{3a^4}{2r^4}\right) \quad (6)$$

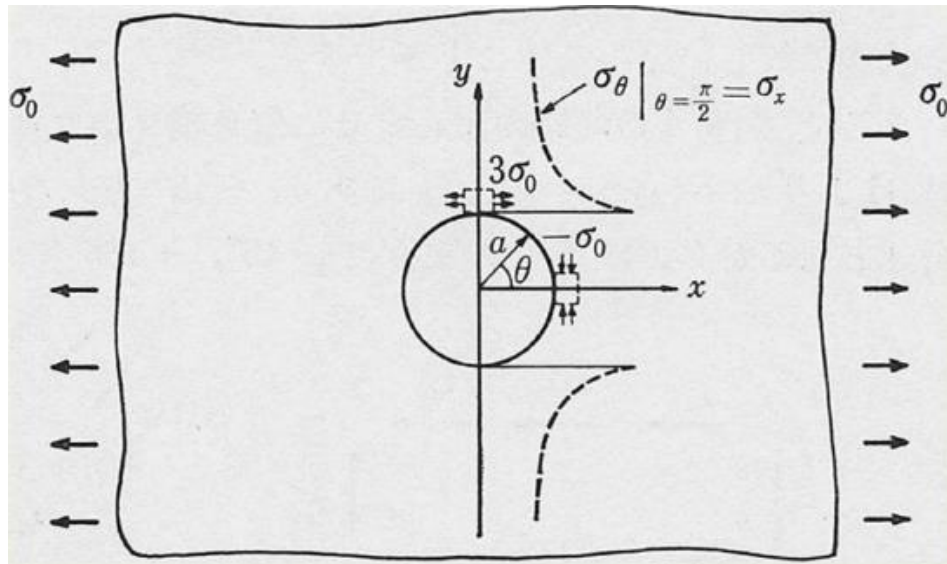


Figure 4. Stress concentration at a circular hole. Results (Murakami n.d., 65)

It is seen, that when calculating for stress concentration factor at “top” point (point C) parameter  $a$  will be equal to hole’s radius  $r$ . When substituting this to equation (6), we will get stress concentration value for points C and D, as the problem is symmetrical, which be equal to 3.

When  $\theta = 0$  the stress concentration value at point B (and A, because of symmetry) can be calculated. When substituting needed values to equation (5b), we will get the result, which equals to -1. (Kirsch 1898, 797–807.)

So final results for circular ellipse problem are  $K_t^{A,B} = -1$  and  $K_t^{C,D} = 3$ . It means that the maximum stress concentration value is 3, therefore, drilling a round hole in a plate far enough from the edges, will make the plate three times weaker. (Murakami n.d., 62-65.)

### 2.2.1 Rule of Ellipse

The rule of ellipse exists in the theory of elasticity, which is used to approximate different complicated problems with cracks, few holes or crack and hole combinations. In figure 5, the general ellipse problem is shown.

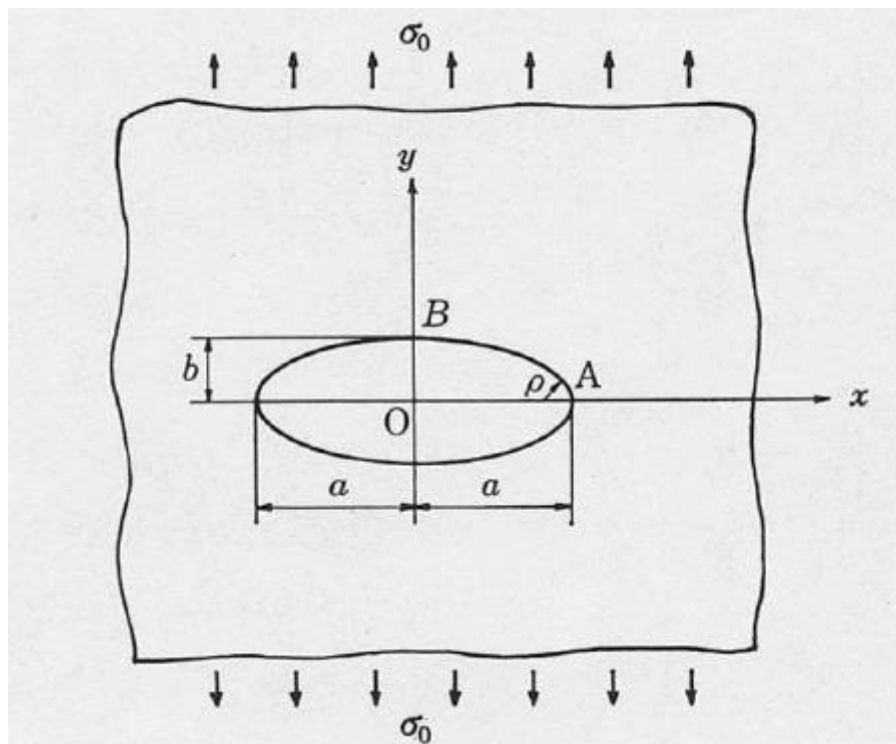


Figure 5. Stress concentration at an elliptical hole (Murakami n.d., 67)

As it was discussed in previous section, maximum stress concentration factor will occur at point A and will be equal to 3 in the case of round hole. In this problem external stress is applied in the y direction, this is important to notice, as many confusions may appear when not being attentive enough with coordinate systems.

From equation (6) generalized formula for the ellipse can be derived



$$\sigma_Y = \sigma_0 \left(1 + \frac{2a}{b}\right) = K_t \sigma_0 \quad (7)$$

From this equation, general formula for stress concentration factor for the ellipse can be easily derived

$$K_t = 1 + \frac{2a}{b} = 1 + 2\sqrt{\frac{t}{\rho}} \quad (8)$$

where  $\rho$  is the root radius of the elliptical hole at the point A and  $t = a$ .

$$\rho = \frac{b^2}{a} \quad (9)$$

$\sigma_X$  at point B, as it was derived before, does not depend on the root ratio.

$$\sigma_X = -\sigma_0 \quad (10)$$

The idea of the ellipse rule, or the concept of equivalent ellipse, is to keep the root radius of a hole or notch at the point equal to approximated ellipse (dotted line in fig. 6). (Murakami n.d., 68-69.)

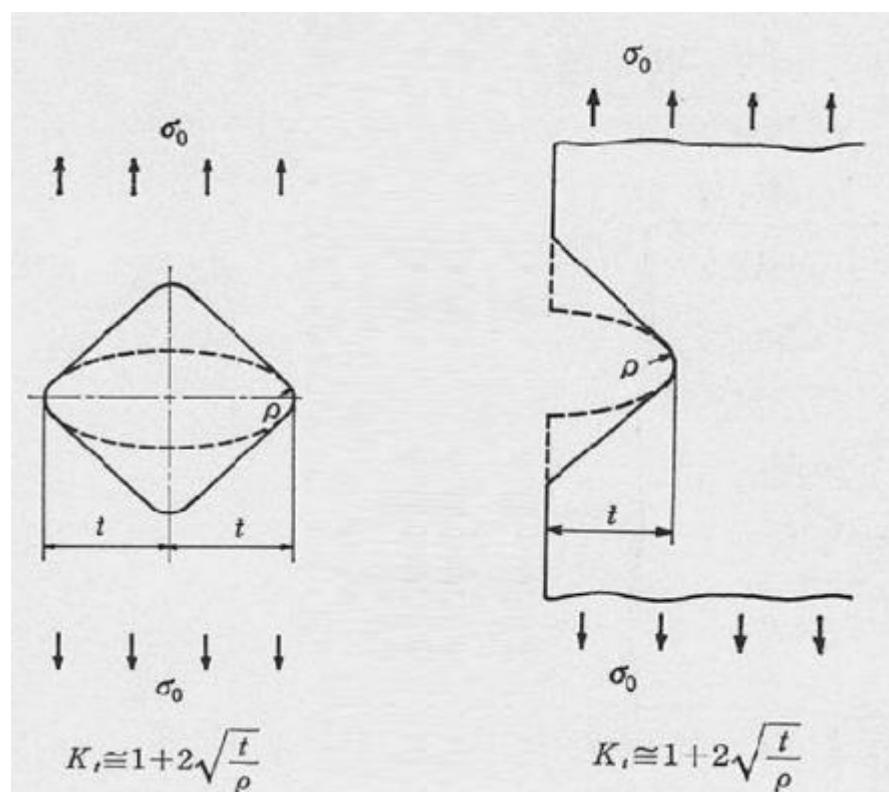


Figure 6. The concept of equivalent ellipse (Murakami n.d., 69)

#### 2.2.1.1. Circle as a special case of ellipse

As it is known from geometry, a circle is a particular case of an ellipse, as  $a = b$  in this case. From equation (7) or (8), if we substitute  $a$  to  $b$  or vice

versa, we will get same results as in the section 2.2 (which are -1 and 3), which proves the values of the stress concentration around the round hole.

### 2.3 Body Force Method

There are some different methods of how to approximate or calculate stress concentration or stress intensity (used for cracks) factor, especially for easy geometries, but sometimes because of geometry approximation is too big or too complex finite element analysis must be used, which can be very inconvenient and time consuming.

In 1957, new approach was proposed by British engineer Eshelby. In 1967 Japanese professor Nisitani developed this method to the form which is used now. It is called body force method. Idea of this method is to use stress field values, which are derived from point forces acting in an infinite plate or body. Boundary conditions are satisfied by continuously embedded point forces along the edges of notches or cracks or along the geometry of a hole. The method works both in two and three-dimensional problems. (Fraga & Hewitt 1983, 1.)

Shortly, the basic idea is to consider a plate with a hole as a plate without hole, in which along the boundary of imaginary hole point forces will be located (fig. 8). These point forces will affect to the plate's stress and strain same way as a hole in a plate without these forces. When solving problem, it is much easier to use these point forces and it is not a difficult task to find numerically how these point forces affect the stress and strain.

#### 2.3.1 Principles of body force method

Let us consider the problem as shown in figure 7.

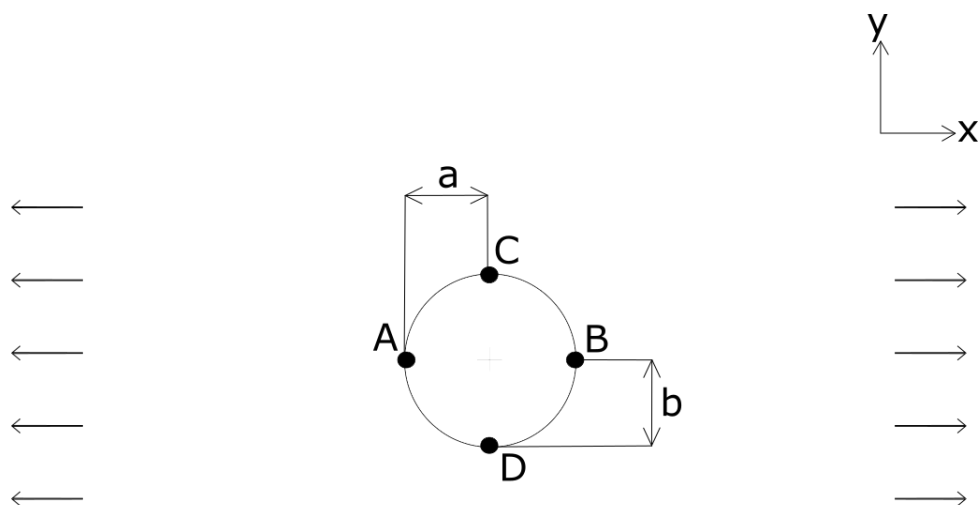


Figure 7. The problem of an elliptical hole in an infinite plate

Stress, applied at infinity, will be marked as  $\sigma_X^\infty$ . From Theory of Elasticity (Timoshenko & Goodier 1951) the x and y components of displacement,  $u$  and  $v$  respectively, at any arbitrary point on the edge of the hole with coordinates  $(\xi;\eta)$  are given by following equation.

$$u = \frac{\sigma_X^\infty}{E} \left(1 + \frac{2b}{a}\right) \xi \quad (11a)$$

and

$$v = -\frac{\sigma_X^\infty}{E} \eta \quad (11b)$$

where  $E$  is the modulus of elasticity (known as Young's modulus) and  $a$  and  $b$  are semi-minor and semi-major widths of the ellipse respectively, as seen in figure 7. In our case we are more interested in round geometry, so  $a$  and  $b$  are equal to each other.

Now to proceed with point forces an elliptical plate should be considered. This plate's dimensions and shape should be same as the hole in previous case. To satisfy boundary conditions in future equations, the displacements of the edges of the elliptical plate must be same as the displacements of the edge of the holes. Then strain components at an arbitrary point on the plate can be derived

$$\varepsilon_X = \frac{\partial u}{\partial \xi} = \frac{\sigma_X^\infty}{E} \left(1 + \frac{2b}{a}\right) \quad (12a)$$

$$\varepsilon_Y = \frac{\partial v}{\partial \eta} = \frac{-\sigma_X^\infty}{E} \quad (12b)$$

$$\gamma_{XY} = 0 \quad (12c)$$

where  $\varepsilon$  and  $\gamma$  are the normal and shear strains respectively.

Corresponding normal and shear stress can be found from Hooke's law

$$\sigma_X = \frac{E}{1-\nu^2} (\varepsilon_X + \nu\varepsilon_Y) = \frac{\sigma_X^\infty}{1-\nu^2} \left(1 + \frac{2b}{a} - \nu\right) \quad (13a)$$

$$\sigma_Y = \frac{E}{1-\nu^2} (\varepsilon_Y + \nu\varepsilon_X) = \frac{-\sigma_X^\infty}{1-\nu^2} \left(1 - \nu\left(1 + \frac{2b}{a}\right)\right) \quad (13b)$$

$$\tau_{XY} = G\gamma_{XY} = 0 \quad (13c)$$

where  $\sigma$  and  $\tau$  are the normal and shear stresses respectively.

Now it can be considered that this elliptical plate, which has these surface stresses, is inserted exactly into the infinite plate from first case with the hole under the external loading with stress equaled to  $\sigma_X^\infty$ . As a result, an infinite plate without a hole is got. As we satisfied boundary conditions for the elliptical hole in equation set (12), it can be claimed, that the problem

of an infinite plate with a hole and external loading is equivalent to the problem of an infinite plate without a hole and with both external stress and stress applied along the imaginary boundary of that ellipse which was defined in equation set (13). These are stresses, which come exactly from continuously embedded point forces along the boundary of the plate, which are the key of body force method. The basic background of that method is shown in figure 8.

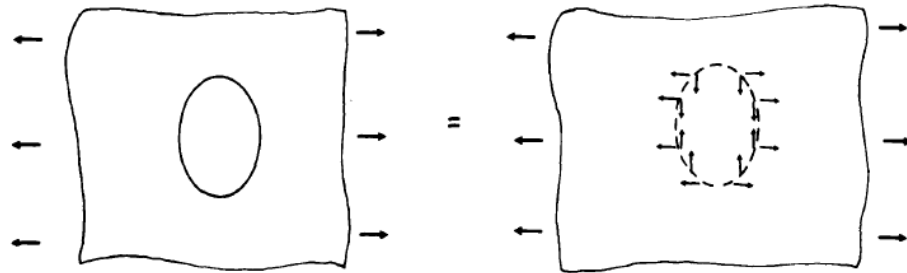


Figure 8. Body force method idea (Fraga & Hewitt 1983, 25)

The next step is finding densities of these body forces in the  $x$  and  $y$  directions,  $\rho_X$  and  $\rho_Y$  respectively. This can be done using equation set (13).

$$\rho_X = \frac{\sigma_X^\infty}{1-\nu^2} \left( 1 + \frac{2b}{a} - \nu \right) \quad (14a)$$

$$\rho_Y = \frac{-\sigma_X^\infty}{1-\nu^2} \left( 1 - \nu \left( 1 + \frac{2b}{a} \right) \right) \quad (14b)$$

where “unit length” is measured in  $y$  direction for  $\rho_X$  and in  $x$  direction for  $\rho_Y$ . (Fraga & Hewitt 1983, 2.)

By Nisitani (1978) Poisson's ratio  $\nu$  does not affect the final results, so it can be set to zero and then the body force densities' equations from set (14) can be written in the following way

$$\rho_X = \sigma_X^\infty \left( 1 + \frac{2b}{a} \right) \quad (15a)$$

$$\rho_Y = -\sigma_X^\infty \quad (15b)$$

Therefore, it can be observed that for an elliptical hole in an infinite plate the densities of body forces are constant along the boundary of the hole. However, in general, body forces themselves will not be constant, so numerical technique must be used to determine them. (Fraga & Hewitt 1983, 3.)

It is seen as well that our resulting densities can be compared with the rule of ellipse used in theory of elasticity. Technically same formulas were

obtained and if round hole case will be considered, where  $a = b$ , then the result will be  $K_t^{A,B} = -1$  and  $K_t^{C,D} = 3$ , and these are values which are got by the rule of ellipse. Therefore, the question might appear why body force method should be used then. For multiple holes cases in an infinite plate or for any other case with holes, cracks, or their combination in an infinite, semi-infinite, or finite plate body force method stays same simple and easy-to-use method for calculating the stresses when other methods' difficultness and the number of system of equations needed for solving problem rises exponentially.

## 2.3.2 Numerical technique

### 2.3.2.1. Formulation

For understanding how the body force method works when obtaining numerical solution, we will stick with the simple problem, which is one elliptical hole in an infinite plate since the results for that case are already known and constant so it can be easily checked whether numerical solution works fine.

The problem shown in figure 7 will be considered. The first step is to divide the boundary of the ellipse to  $MM$  equal segments. The body force densities  $\rho_x$  and  $\rho_y$  are set to constant value for each segment. The whole idea of body force method is to calculate the influence coefficients from the stress field for a point force. Some boundary conditions should be applied, and in our case, it will be the mid-point of each segment to be free from stress. Later, the stress concentration factor and the stress at any arbitrary point of the plate can be calculated from systems of linear equations. (Fraga & Hewitt 1983, 14.)

### 2.3.2.2. Solution sequence

Firstly, as mentioned above, the ellipse is divided into  $MM$  equal intervals. The elliptical parameter  $\varphi$ , which comes from the equations of an ellipse, is used in this step.

$$\xi = a \cos\varphi \quad (16a)$$

$$\eta = b \sin\varphi \quad (16b)$$

It is easy to derive the  $\varphi$  values at the end points and mid-points of the each segment. Let  $N$  to be the index number of segments. Then at the end points values are

$$\varphi_{N-1} = -\frac{\pi}{2} + \frac{2(N-1)\pi}{MM} \quad (17a)$$

$$\varphi_N = -\frac{\pi}{2} + \frac{2N\pi}{MM} \quad (17b)$$

and  $\varphi$  value at the mid-point of a segment is

$$\varphi_M = -\frac{\pi}{2} + \frac{(2N-1)\pi}{MM} \quad (17c)$$

The  $x$  and  $y$  values of each segment's mid-point can be derived easily as well. It is important to notice, that in the case with one hole the local coordinate system of the hole and the global coordinate system are coinciding (if not defined otherwise, but this is the most practical way). In cases with two or three holes, it is important to define local coordinate systems and the global coordinate system clearly, so there are no coordinate mistakes during calculations. Usually, the main hole's, which is the central one is most of cases, local coordinate system is coinciding with the global one.

So, the  $x$  and  $y$  values of each segment's mid-point are

$$x = a \cos\left(-\frac{\pi}{2} + \frac{(2N-1)\pi}{MM}\right) \quad (18a)$$

$$y = b \sin\left(-\frac{\pi}{2} + \frac{(2N-1)\pi}{MM}\right) \quad (18b)$$

The second step is to calculate the influence coefficients, which are the stress values at the mid-points of the  $M$ -th segment due to a body force of the  $N$ -th segment. Densities  $\rho_x$  and  $\rho_y$  are considered to be equal 1 each.

Fraga and Hewitt (1983) and Nisitani (1978) formulas for fundamental stress fields are used. The stress fields are needed for calculating the influence coefficients. For plane stress situation, the stress fields at  $(x,y)$  for point forces  $X$  and  $Y$  acting at  $(\xi,\eta)$  for an infinite plate are following

$$\sigma_x^X = -Fl\{(3 + \nu)l^2 + (1 - \nu)m^2\}X \quad (19a)$$

$$\sigma_y^X = Fl\{(1 - \nu)l^2 - (1 + 3\nu)m^2\}X \quad (19b)$$

$$\tau_{xy}^X = -Fm\{(3 + \nu)l^2 + (1 - \nu)m^2\}X \quad (19c)$$

$$\sigma_x^Y = -Fm\{(1 + 3\nu)l^2 - (1 - \nu)m^2\}Y \quad (19d)$$

$$\sigma_y^Y = -Fm\{(1 - \nu)l^2 + (3 + \nu)m^2\}Y \quad (19e)$$

$$\tau_{xy}^Y = -Fl\{(1 - \nu)l^2 + (3 + \nu)m^2\}Y \quad (19f)$$

where

$$l = \frac{x-\xi}{y} \quad (20a)$$

$$m = \frac{y-\eta}{y} \quad (20b)$$

$$F = \frac{1}{4\pi y(l^2+m^2)^2} \quad (20c)$$

As it could be easy to get confused by many different upper and lower indices, it is worth to notify, that upper index, which is written in capital letter, stands for body force's direction, according to which stress field is calculated, whereas lower index stands for stress direction.

Knowing the stress fields' formulas, it is easy to express the needed influence coefficients. As the relations  $d\eta = b \cos\varphi d\varphi$  and  $-d\xi = a \sin\varphi d\varphi$  are true all over the ellipse, the coefficients are derived following way (Nisitani 1978.)

$$\sigma_{xM}^{XN} = \int_N \sigma_x^X(\xi, \eta, x, y)|_{x=1} b \cos\varphi d\varphi \quad (21a)$$

$$\sigma_{yM}^{XN} = \int_N \sigma_y^X(\xi, \eta, x, y)|_{x=1} b \cos\varphi d\varphi \quad (21b)$$

$$\tau_{xyM}^{XN} = \int_N \tau_{xy}^X(\xi, \eta, x, y)|_{x=1} b \cos\varphi d\varphi \quad (21c)$$

$$\sigma_{xM}^{YN} = \int_N \sigma_x^Y(\xi, \eta, x, y)|_{y=1} a \sin\varphi d\varphi \quad (21d)$$

$$\sigma_{yM}^{YN} = \int_N \sigma_y^Y(\xi, \eta, x, y)|_{y=1} a \sin\varphi d\varphi \quad (21e)$$

$$\tau_{xyM}^{YN} = \int_N \tau_{xy}^Y(\xi, \eta, x, y)|_{y=1} a \sin\varphi d\varphi \quad (21f)$$

where  $\int_N$  stands for integration of the  $N$ -th interval.

As it was mentioned above, the influence coefficients are the stress values at the mid-point of the  $M$ -th segment due to a body force on the  $N$ -th segment. This is important to understand so there is no confusion when reading indices, values and choosing correct relevant data for equations.

In some special cases, it is possible to make the whole solution process simpler. For example, in this case (one elliptical round hole), or when having two auxiliary holes at the same distance from the main hole from each side. The key is symmetry. Whenever the problem is symmetrical about the  $x$  and  $y$  axes, the influence coefficients should be calculated only for one quarter of the ellipse. The important moment is that during calculating these, for the chosen quarter all the influence coefficients must be calculated, which means that should be taken into consideration the effects of all body forces all around the ellipse. (Fraga & Hewitt 1983, 15.)

The third step in this solution sequence is to apply above-mentioned conditions to make the mid-point of each segment free from stresses.

$$\sum_{N=1}^{MM} \rho_{xN} (\sigma_{xM}^{XN} \cos\theta + \tau_{xyM}^{XN} \sin\theta) + \sum_{N=1}^{MM} \rho_{yN} (\sigma_{xM}^{YN} \cos\theta + \tau_{xyM}^{YN} \sin\theta) + \sigma_X^\infty \cos\theta = 0 \quad (22a)$$

$$\sum_{N=1}^{MM} \rho_{xN} (\sigma_{yM}^{XN} \sin\theta + \tau_{xyM}^{XN} \cos\theta) + \sum_{N=1}^{MM} \rho_{yN} (\sigma_{yM}^{YN} \sin\theta + \tau_{xyM}^{YN} \cos\theta) = 0 \quad (22b)$$

where  $\rho_{xN}$  and  $\rho_{yN}$  are the body force densities of the  $N$ -th segment,  $\sigma_X^\infty$  is the stress from applied external loading and  $\theta$  is the angle between the  $x$ -axis and the normal to the ellipse at the mid-point of the  $M$ -th interval. (Fraga & Hewitt 1983, 15-16.)

$$\theta = \arctan\left(\frac{a}{b} \tan\varphi_M\right) \quad (23)$$

where  $\varphi_M$  is  $\varphi$  value at the mid-point of a segment, found in equation (8c).

As a result, a system of equations will be got. It will contain  $2MM$  linear equations with  $2MM$  unknowns, which will be  $\rho_{xN}$  and  $\rho_{yN}$ .

The fourth step is to calculate stress at any arbitrary point P of the plate. Knowing the influence coefficients and the body force densities, it is possible to find numerical value of the stress by following equations

$$\sigma_X = \sum_{N=1}^{MM} (\sigma_{xP}^{XN} \rho_{xN} + \sigma_{xP}^{YN} \rho_{yN}) + \sigma_X^\infty \quad (24a)$$

$$\sigma_Y = \sum_{N=1}^{MM} (\sigma_{yP}^{XN} \rho_{xN} + \sigma_{yP}^{YN} \rho_{yN}) \quad (24b)$$

$$\tau_{XY} = \sum_{N=1}^{MM} (\tau_{xyP}^{XN} \rho_{xN} + \tau_{xyP}^{YN} \rho_{yN}) \quad (24c)$$

The fifth step is to evaluate actual stress concentration factors, which are the answers actually needed and searched during using the body force method. As it comes from the definition of stress concentration and equation (1), for the problem shown in figure 7 the stress concentration factor can be calculated following way

$$K_{TA} = \frac{\sigma_X(at A)}{\sigma_X^\infty} \quad (25a)$$

$$K_{TD} = \frac{\sigma_X(at D)}{\sigma_X^\infty} \quad (25b)$$

As the body force densities around the ellipse are constant, as mentioned above, there will be no error for this problem whatever value for  $MM$  is chosen, so the stress concentration factor will be the same regardless of



the number of segments. However, the  $MM$  value will affect when solving two, three or more holes problems – the smaller will be amount of segments ( $MM$  value), the less accurate will be the stress concentration values.

### 2.3.2.3. Singularity

In the numerical integration, a singularity will appear in the stress field equations (19) at  $y = \eta$  and  $x = \xi$ . It is happening when the stress are evaluated at mid-point  $(x,y)$  of a segment due to a point force  $X$  or  $Y$  of the same segment. It does not affect the integration over all other segments.

To integrate over the singularity, the integral is splitted to three disjoint subsegments

$$[\varphi_{N-1}, \varphi_{MID} - \epsilon]; [\varphi_{MID} - \epsilon, \varphi_{MID} + \epsilon]; [\varphi_{MID} + \epsilon, \varphi_N] \quad (26)$$

where the  $x$  and  $y$  are coordinates of segment's mid-point.

$$x = a \cos \varphi_{MID}; y = b \sin \varphi_{MID} \quad (27)$$

The first and last subintervals are straightforward, as they do not contain the midpoint of the segment, which is the cause of singularity.

The second subsegment can be approximated by a function, which will not contain a singularity. For that  $\epsilon$  should be chosen such a way, that it will be small enough relatively to the length of the interval  $[\varphi_{N-1}, \varphi_N]$ .

The integrals over the singularities for the ellipse can be approximated by the following simple analytic expressions (Nisitani, Suematsu & Saito 1973)

$$\Delta \sigma_{xM}^{XM} = \epsilon \frac{k}{4\pi(k^2+w^2)^3} [6w^6 + (3 - 2k^2)w^4 - 6k^2w^2 - k^4] \quad (28a)$$

$$\Delta \sigma_{yM}^{XM} = \epsilon \frac{k}{4\pi(k^2+w^2)^3} [-2w^6 - (1 - 6k^2)w^4 + 6k^2w^2 - k^4] \quad (28b)$$

$$\Delta \tau_{xyM}^{XM} = \epsilon \frac{k}{4\pi(k^2+w^2)^3} [-9w^5 - (6 + 2k^2)w^3 + k^2(2 - k^2)w] \quad (28c)$$

$$\Delta \sigma_{xM}^{YM} = \epsilon \frac{k}{4\pi(k^2+w^2)^3} [-w^6 + 6k^2w^4 + k^2(6 - k^2)w^2 - 2k^4] \quad (28d)$$

$$\Delta \sigma_{yM}^{YM} = \epsilon \frac{k}{4\pi(k^2+w^2)^3} [-w^6 - 6k^2w^4 + k^2(2 - k^2)w^2 + 6k^4] \quad (28e)$$

$$\Delta \tau_{xyM}^{YM} = \epsilon \frac{k}{4\pi(k^2+w^2)^3} [(1 - 2k^2)w^5 + 2k^2(1 + 3k^2)w^3 + 9k^4w] \quad (28f)$$

where

$$k = \frac{b}{a} ; w = \tan\varphi_M \quad (29)$$

For this  $M$ -th interval when  $y = \eta$  and  $x = \xi$ , additional stress, which are the stresses infinitesimally close to the ellipse subjected to a body force, must be added to  $\sigma_{xM}^{XM}, \sigma_{xM}^{YM}, \sigma_{yM}^{XM}, \sigma_{yM}^{YM}, \tau_{xyM}^{XM}, \tau_{xyM}^{YM}$  respectively.

The final equations of these stresses are following (Fraga & Hewitt 1983, Appendix 5.)

$$\Delta\sigma_x^X \Big|_{\rho_X=1} = -\frac{1}{16}(5 + 4 \cos 2\theta - \cos 4\theta) \quad (30a)$$

$$\Delta\sigma_y^X \Big|_{\rho_X=1} = \frac{1}{16}(1 - \cos 4\theta) \quad (30b)$$

$$\Delta\tau_{xy}^X \Big|_{\rho_X=1} = -\frac{1}{16}(2 \sin 2\theta - \sin 4\theta) \quad (30c)$$

$$\Delta\sigma_x^Y \Big|_{\rho_Y=1} = \frac{1}{16}(1 - \cos 4\theta) \quad (30d)$$

$$\Delta\sigma_y^Y \Big|_{\rho_Y=1} = -\frac{1}{16}(5 - 4 \cos 2\theta - \cos 4\theta) \quad (30e)$$

$$\Delta\tau_{xy}^Y \Big|_{\rho_Y=1} = -\frac{1}{16}(2 \sin 2\theta + \sin 4\theta) \quad (30f)$$

As it was mentioned above,  $\vartheta$  is the angle between the x-axis and the normal to the ellipse at the mid-point of the  $M$ -th interval. (Fraga & Hewitt 1983, 16-17.)

#### 2.3.2.4. Results

In table 1 the body force densities  $\rho_X$  and  $\rho_Y$  for each segment of a circular elliptical single hole in an infinite plate is shown.

As it is seen, the results are same as in section 2.3.1. As it was mentioned above, the body force densities  $\rho_X$  and  $\rho_Y$  are constant all around the ellipse. As it resulted from equations (15),  $\rho_X = 3$  and  $\rho_Y = -1$ .

Table 1. Body force densities for circle in infinite plate (Fraga & Hewitt 1983, 18.)

Segment	$\rho_X$	$\rho_Y$
1	3.000000	-1.000000
2	3.000000	-1.000000
3	3.000000	-1.000000
4	3.000000	-1.000000
5	3.000000	-1.000000
6	3.000000	-1.000000
7	3.000000	-1.000000
8	3.000000	-1.000000

### 3 INTERACTION OF STRESS CONCENTRATION

#### 3.1 Matlab analysis

It is possible to find any plate's stress concentration, when there is an elliptical hole or few holes combination. There are some guides existing, how to evaluate stress concentration, even using computers, but most of them are a little bit out-of-date or do not provide detailed information, just giving some values and some relations. Idea of this work is to come up with a Matlab code, which later will be accessible for public use. This code will give the stress concentration factors for boundaries of all holes present in the plate. The only thing, which is needed to be input, is the size of the main and auxiliary holes and distance between holes. We will stick with two cases: one auxiliary hole and two auxiliary holes, which will be located symmetrically about the main hole.

First, using Nisitani's body force method, discussed in the section 2.3.1 and 2.3.2, the code for one hole should be programmed, so the results can be easily checked with manual calculated ones to be sure that the code is working well. Later, the code will be modified, as body force method's approach will not change, because the only thing, which should be added – coordinates of other holes.

##### 3.1.1 An infinite plate with one drilled hole

The whole code is available in the Appendix 1. The input of the function is following

- $MM$ : number of segments;
- $a$  and  $b$ : semi-minor and semi-major widths of the ellipse;
- $\epsilon$ : the step, which is used for dividing segment to subsegments in order to integrate over the singularity.

Some minor details should be mentioned about some of the input values.

In Fraga & Hewitt 1983 all the shown results for different cases, which are cracks, holes, combination of cracks and holes in infinite and semi-infinite plates, are shown for some different segment numbers  $MM$ . All of them are factor of four. There is no explanation unfortunately, as it might be mentioned in original papers of Nisitani, which are available only in Japanese. It was discovered after actually trying different values of  $MM$  in the code. First, the conclusion was made, that only even numbers should be used, as by some reason odd numbers were giving weird results. Later, as it is used in Nisitani's results,  $MM$  value was narrowed up to numbers, which are factor of four. In further results, everywhere  $MM = 4$  value was used.

For *epsilon* there are no special rules, it was chosen in experimental way, so it does not provide a big error and still works for integrating over singularity. To get the results in the program *epsilon* = 0.000002 was used.

Small analysis of code will be shown to explain a little bit how it works, so it is easy to use the code.

First, important thing is in Matlab should be the folder selected, which contains the program file. It is done using the button on the interface, shown in figure 9.

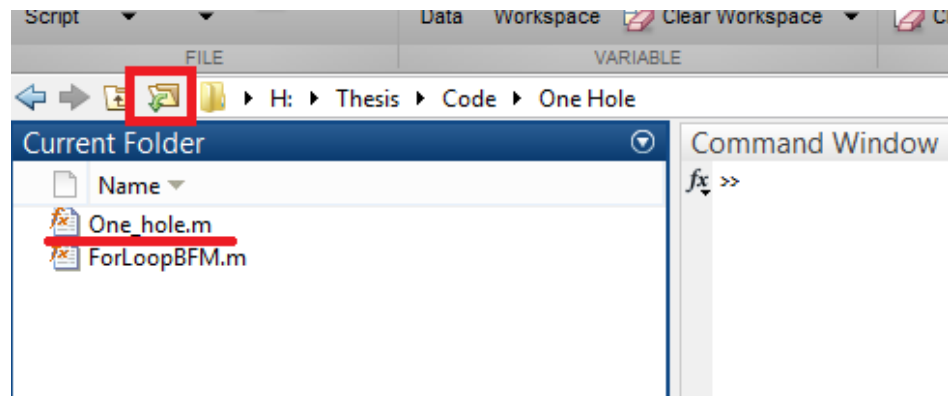


Figure 9. Matlab interface. Running the program. Step 1

After that needed function should be opened, in this case it is "One\_hole.m". This is done to input needed values. Input values are changed in two steps:

- in the interface of program (shown in figure 10);
- before running program itself in main interface (shown in figure 11).

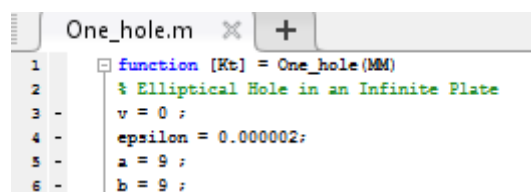


Figure 10. Matlab interface. Running the program. Step 2

The following lines should be changed accordingly to the data

```
epsilon = 0.000002;
a = 9;
b = 9;
```

Poisson's ratio  $\nu$  should not be changed, as in section 2.3.1 it was discussed that Poisson's ratio has no effect on the stress concentration factor.

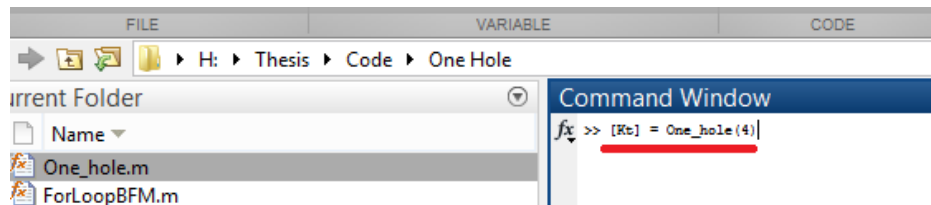


Figure 11. Matlab interface. Running the program. Step 3

After that in Matlab command window should be written the function itself, which is first line of the code (figure 10). In round brackets there are located variables, which should be input. In our case, as it is seen, it is only amount of segments  $MM$ .

$[Kt] = \text{One\_hole}(MM)$

As it was discussed above, we use  $MM = 4$ , putting the value in the round brackets. The results will appear below the input function, as it is shown in figure 12. In the left bottom corner maximum and minimum values of the resulting array are shown, as well as array itself.

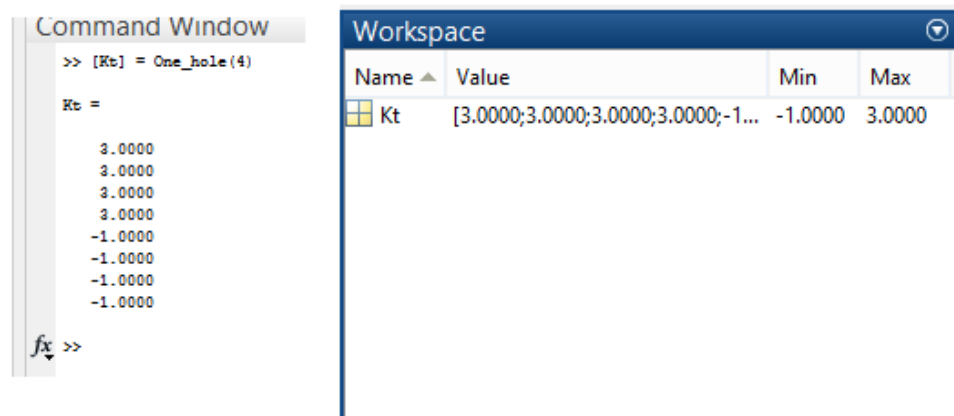


Figure 12. Matlab interface. Results

As it is seen, the results are same as calculated by different methods, mentioned above, including, theory of elasticity and manual body force method.

In other programs for more difficult cases other inputs will be used, but this will be discussed in following chapters and sections.

Two useful Matlab commands to use are

clear all  
clc

First one is clearing the results from the workspace, second one – cleaning the command window.

Now the short explanation of the code itself.

```
ellip = zeros(MM,2);
ellip_M = zeros(MM,1);
```

```
coord = zeros(MM,4);
coord_M = zeros(MM,2);
```

In these lines, empty matrices for storing coordinates are created. “Zeros” means that matrix will be full of zeroes, and matrix will contain  $MM$  lines and two columns, in case of variable “ellip”.

In table 2 it is shown what every column of these four above mentioned variables means, whereas lines stand for each segment.

Table 2. Coordinate storing in Matlab

	Column 1	Column 2	Column 3	Column 4
ellip	Start $\varphi$ -coord. for start point of segment	End $\varphi$ -coord. for start point of segment		
ellip_m	$\varphi$ -coord. for mid-point of segment			
coord	Start x-coord. for start point of segment	Start y-coord. for start point of segment	End x-coord. for start point of segment	End y-coord. for start point of segment
coord_m	x-coord. for mid-point of segment	y-coord. for mid-point of segment		

The following code will be analyzed without checking all code, as many lines are very similar, for example, coordinate calculation for the x and y direction are almost same. In order to make it more compact, it will be mentioned only one code’s line will be shown here.

```
sigmainf1 = ones(MM,1);
sigmainf2 = zeros(MM,1);
```

External stress is defined in these two lines for each segment, as later it will be used for calculating stress concentration factor. As the problem with axial tension is considered, in other axial direction stress will be zero. The external stress value is considered to be one, as the stress concentration is the ratio of the stresses.

```

for N=1:MM
...
end

```

This block means that the inside functions will run in a loop for each  $N$  value, which is segment's order number. In the loop there are equations (17) and (18) for calculating the coordinates.

```
Sigmax_X = zeros(MM,MM);
```

Empty matrices for storing the values of influence coefficients are created. As for each segment there are  $MM$  influence coefficients, the size of each matrix will be  $MM*MM$ . But firstly, fundamental stress fields should be calculated.

```
x0 = @(q) a.*cos(q);
```

```
l = @(q)(x-x0(q))./y;
```

```
m = @(q)(y-y0(q))./y;
```

```
F = @(q)1./(4.*pi.*y.*(l(q).^2+m(q).^2).^2);
```

Symbol @ in Matlab is used for transforming variable to function of some variable.

$x_0$  and  $y_0$  are the  $\xi$  and  $\eta$  from equations (20).  $l$ ,  $m$ , and  $F$  are defined in equations set (20).

```
Sigmax_X_sum = @(q)
```

These functions are the fundamental stress fields, which are presented in equations (19).

```

if H ~= N
...
elseif H == N
...
end

```

This block is dealing with singularity case.  $N$  stands for segment's order number, for which influence coefficients are calculated and  $H$  – for the segment's order number, according to which segment's body force influence coefficients are calculated.

As it was mentioned in section 2.3.2.3, singularity occurs when influence coefficient of a segment is calculated due to same segment's body force. So "if" part is very simple as there is no singularity.

```
Sigmax_X(N,H) = quad(Sigmax_X_sum,ellip(H,1),ellip(H,2));
```

This is the step used after “if” statement, it is a regular integration, given in equations (21). “quad” command stands for integration. First element in brackets is an integrand, next element is start point of integration and last element – end point of integration.

```
k=b/a ;
w=tan(ellip_M(H,1));
t(N,1) =atan((a/b)*tan(ellip_M(H,1)));
ADD_1
```

The following code is for “elseif” statement. Equations (29) are given first, after that – ADD\_\* functions are calculated, which are equations (28).

```
Sigmax_X(N,H)
```

The influence coefficients for singularity case are calculated, summing up three subsegments’ values, which are given in formula (26).

```
C_theta = zeros(MM,1);
S_theta = zeros(MM,1);
```

Empty matrices for storing cosine and sine values of  $\vartheta$  for each segment, which are used in equations (23). Itself  $\vartheta$  is defined in equation (22). In the following lines the value for the  $\vartheta$  and its cosine and sine are calculated.

```
Sigma_X1 = zeros(MM,MM);
```

This variable and analogic ones in following three lines are part of equations (22). If to be exact – the parts in round brackets respectively. Following four lines after “for” statements are the actual calculations of these variables.

```
sigma_inf = [sigmainf1;sigmainf2];
SIGMA1 = [Sigma_X1,Sigma_X2];
SIGMA2 = [Sigma_Y1,Sigma_Y2];
SIGMA = [SIGMA1;SIGMA2];
```

These four lines are written for composing the results. When comma is used, two matrices are combined, adding more columns, when semicolon is used – lines are added. Figure 13 is representing visually how it works.



$$A = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 8 & 9 \end{bmatrix} \quad C = [A;B] \\ D = [A,B]$$

$$C = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ 1 & 3 \\ 8 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 4 & 1 & 3 \\ 5 & 7 & 8 & 9 \end{bmatrix}$$

Figure 13. Combining of matrices in Matlab

`Kt = SIGMA \sigma_inf`

The final line of the program and the thing, which is needed to find out in the program and the problem itself – the stress concentration factor, which is given in equation (1).

### 3.1.2 An infinite plate with two or three drilled holes

As it was mentioned above, code for both needed cases is almost the same; the one difference will be in one of the inputs. The whole code is available in appendices 2-6.

One more difference is that program for these cases is written in modules for making the whole process more understandable. The module subprograms should be saved in the same folder as the main programs, as subprograms are called during calculation process and are run automatically.

Firstly, the main programs will be shortly described with specifying inputs, and then – subprograms.

`function[Kt] = Two_Holes(MM,s1,s2,a1,b1,a2,b2)`

This is first line from program “Two\_Holes.m” and, as it was mentioned above, in round bracket are the inputs, which should be put by a user. Figure 14 shows what every variable stands for, except the *MM*, which is the number of segments.

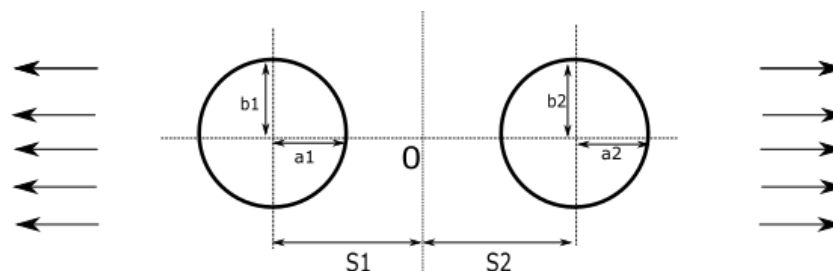


Figure 14. Inputs for the Matlab program “Two\_Holes.m”

Everything other is working same way as in the program for one hole, except the fact, that some parts from the program are brought out to subprograms. It is important to notice, that  $s1$  is located on left from the  $y$ -axis of the global coordinate system. It means, that in Matlab  $s1$  value must be negative.

```
function[Kt] = Three_Holes(MM,s1,s2,s3,a1,b1,a2,b2,a3,b3)
```

This is the first line with inputs of the program for three holes problems. Figure 15 shows the explanation of variables.

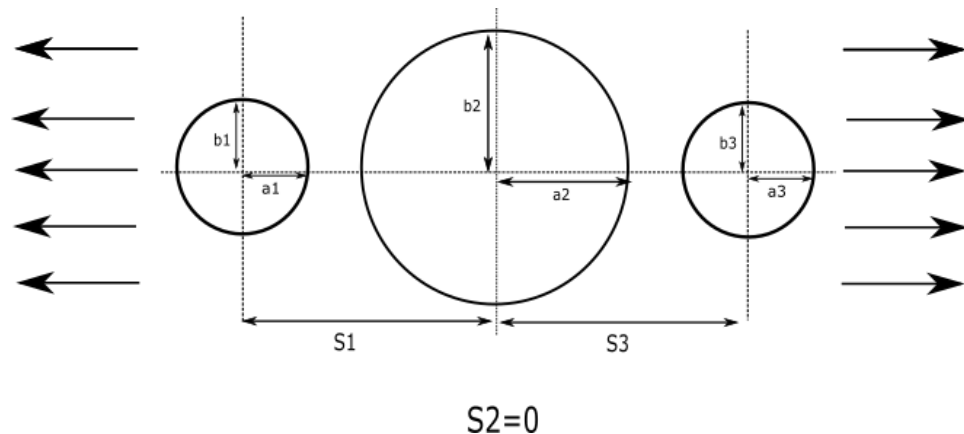


Figure 15. Inputs for the Matlab program “Three\_Holes.m”

Same as in the case with “Two\_Holes.m”,  $s1$  value must be negative, whereas  $s2$  value always equals zero, as it is shown in the picture.

Subprograms “Module” and “Module\_SP” contain program code for calculating the stresses from all the integrations and systems of equations, discussed in section 2.3.2. The difference between these two modules are in integrating over the singularity. Sub-program “ellip\_MidPoint” is used for calculating the coordinates of segments. (Otto & Denier 2005.)

### 3.2 Preliminary results for a plate with two and three holes

For the problem with two holes only horizontal positioned case was checked, which is shown in figure 16.

Vertical location of holes will not be checked because stress concentration factor will be much higher. It can be checked, using the rule of equivalent ellipse, as well as it can be seen from the schematic view of the stress flow, which was discussed in section 2.1. The stress flow of the plate with two vertically located holes is shown in figure 17.

Rule of ellipse was discussed in section 2.2.1. Equation (8) is used to evaluate the stress concentration value. Figures 18 and 19 show the equivalent ellipse, drawn around two holes.

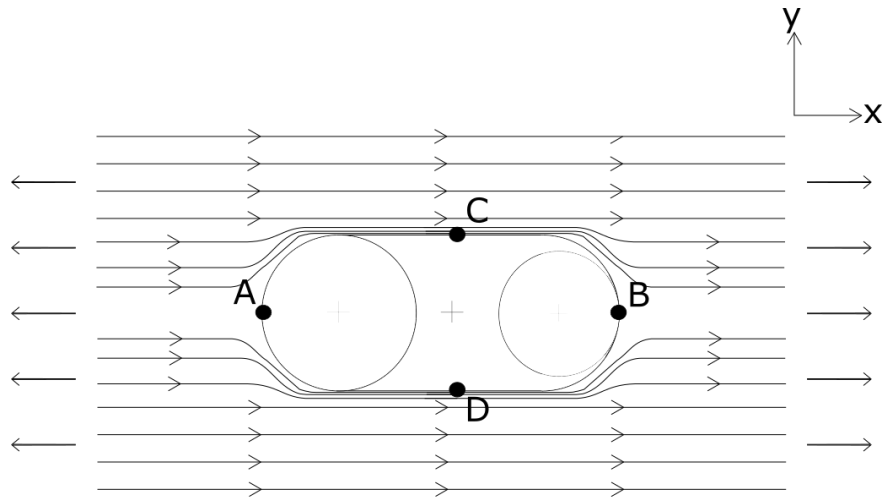


Figure 16. Problem with two horizontally located holes

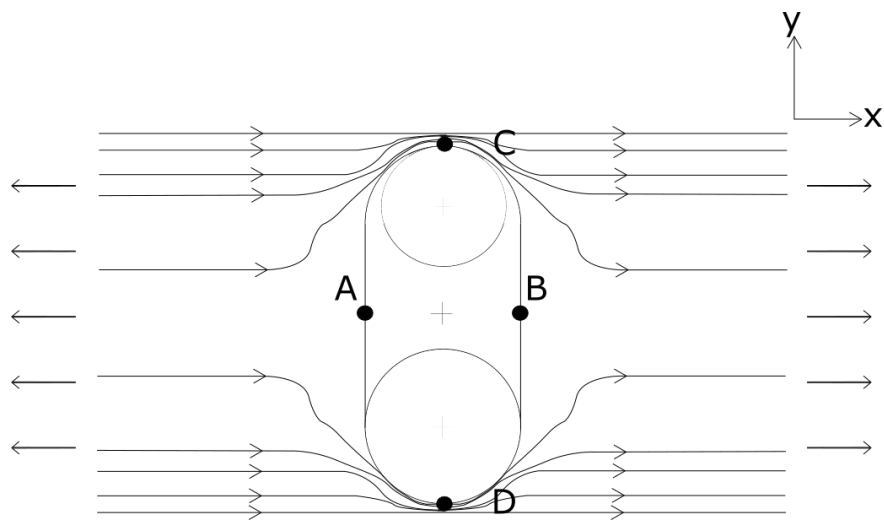


Figure 17. Problem with two vertically located holes

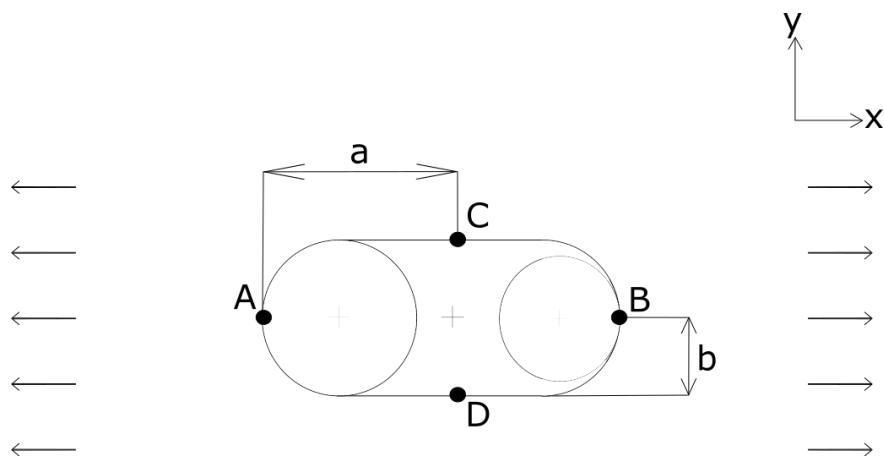


Figure 18. Equivalent ellipse in the problem with two horizontally located holes

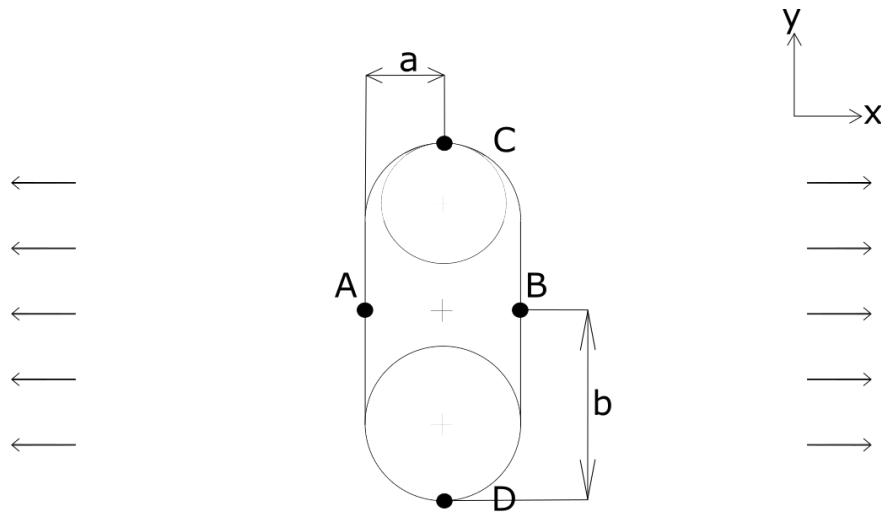


Figure 19. Equivalent ellipse in the problem with two vertically located holes

Equation (8) is based on the figure's 5 drawing and definition of variables. After customizing the equation for the above-mentioned case, we get the following

$$K_t = 1 + \frac{2b}{a} \quad (31)$$

Let us consider that sizes of big holes and small holes in both cases are same, so equivalent ellipse's are of the same size, the only difference in these problems is that ellipse is rotated by  $90^\circ$ .

In the problem with horizontal location of holes we assume that  $a = 2$  and  $b = 1$  and in the problem with vertical location –  $a = 1$  and  $b = 2$ . Such dimensions are satisfying our condition of equality of the ellipses.

Table 3 shows the result for stress concentration factor for these two problems, using equation (31).

Table 3. Results of problems with two holes

Case	Horizontal location	Vertical location
$K_t$	2	5

From these results, it is clearly seen why no plates with vertically located holes are observed and analyzed, as this case will have much higher stress concentration value, and this is not our target.

For preliminary analysis in Matlab it was decided to use ratios  $\frac{R_1}{R_2}$  and  $\frac{R_1}{e}$  changeable from 0.1 to 1 and keeping  $R_1$  constant, so the results will be able to be used for different kind of situations as they will be relative.  $R_1$  and  $R_2$  are radii of big and small holes respectively and  $e$  –

distance from the edge of left hole to the edge of right hole. Figure 20 shows these dimensions, which are used in the results and appendices.

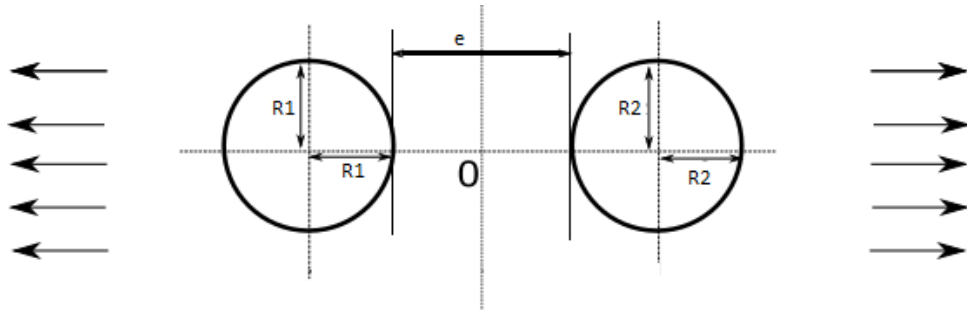


Figure 20. Dimensions used in the results

As an example, the results of once case will be shown in this section (table 4). Full relative, as mentioned above, results for the problem with holes, are available in appendix 7 and 8.

Table 4. Stress concentration factor for two circular holes in infinite plate.  $MM=4$

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.09	3.04	2.99	2.95	2.92	2.90	2.88	2.86	2.85	2.84	2.83

## 4 FINITE ELEMENT METHOD

### 4.1 Finite element method background

The development of the finite element analysis began in the middle of 20<sup>th</sup> century. The finite method analysis (FEA), or the finite element method (FEM), is used for numerical solution of field problems. The main idea of FEM is to discretize continuum to finite elements, which are small pieces of a structure. They are called “finite” in opposite to infinitesimal elements that are used in calculus. These elements are connected with each other at points that are called nodes (fig. 21). The arrangement of the elements is called a mesh, therefore the process of discretization of continuum to the elements is called meshing. In order to get the solution, a system of algebraic equation should be solved for unknown at nodes. These nodal unknowns are values of the field quantity. Combining nodal solutions for one element together with the field will determine the spatial variation of the field in the element. Hence the field quantity over the whole structure is the approximated element by element. From that the conclusion can be made that an FEA solution is approximate, not exact, but still it is very powerful method, and some modifications during the analysis can be made to improve the accuracy of the result. FEM itself has many advantages comparing to other numerical methods:

- FEM can be used in any field problem: stress analysis, heat transfer, magnetic fields, etc.;
- No limitations for geometrical shapes of the structure;
- Any kind of boundary and loading conditions can be used;
- Anisotropy within the body or even within the element can be used;
- Different engineering components can be combined + bars, beams, plates, cables, wires, etc.;
- Mesh can be improved by an engineer manually where needed (i.e., in stress concentration points) to improve the approximation.

The FEA is done in several steps, which are the following

1. Problem classification;
2. Mathematical model;
3. Discretization;
4. Numerical analysis,
5. Interpretation of results.

Sections 4.1.1 to 4.1.4 discuss on these steps more thoroughly. (Cook, Malkus, Plesha & Witt, R. J. 2001, 1-2.)

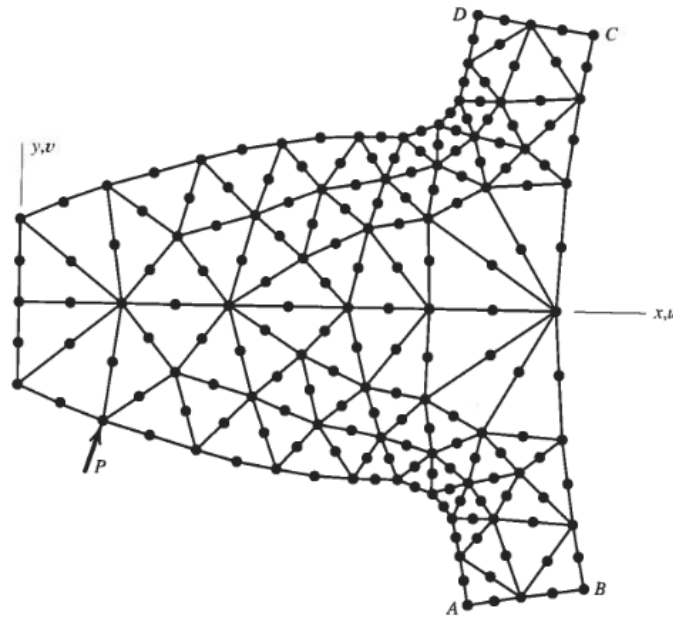


Figure 21. A two-dimensional model. Elements, nodes, and loading is shown (Cook et al. 2001, 2)

#### 4.1.1 Problem classification and modelling

This is the first step in any engineering problem, not only in using FEM. The loading conditions, time dependence, degree of freedoms, supports – everything should be clearly defined. After the physical nature of the problem is well understood, the next step can be started, which is creating a mathematical model.

In modelling, all essential features must be included, but unnecessary details can be dropped out, so that further analysis will not be too complicated yet not too approximate. To transfer from geometrical model to mathematical model differential equations, that describe the behavior of the structure, and boundary conditions are introduced. As FEA is a simulation, not a reality, it is important to make a mathematical model appropriate and adequate, otherwise the results will not be reliable.

Geometry, material properties, loads, boundary conditions are simplified in a mathematical model, and these simplifications are based on engineer's understanding of the problem, which again highlights the importance of the correct problem classification. Some examples of these simplifications are

- Material may be regarded isotropic, homogeneous and linearly elastic (although common materials are not like this);
- A load distributed over a relatively small area can be considered as a force concentrated at a point (which is not physically true);

- Support can be considered as fixed one (although no support is completely rigid) and so on.

After a mathematical model is done, the transformation to the finite element model, also called as a discretization, can be performed. (Cook et al. 2001, 3.)

#### 4.1.2 Discretization

A mathematical model is split to a number of finite elements, which is also called meshing. In this step a continuum is represented by a piecewise continuum that is defined by a finite number of nodal quantities, and simple interpolation within each element is done. It is another approximation used in FEM. It is important to understand that by this step two sources of error are introduced: modelling error and discretization error. Those two errors can be reduced by improving the model and by using larger number of elements, for example, by mesh refinement (i.e. h-method), use of higher order elements (i.e. p-method) and so on (Vavilov 2016).

A simple case of discretization of two-dimensional model is shown in figure 22.

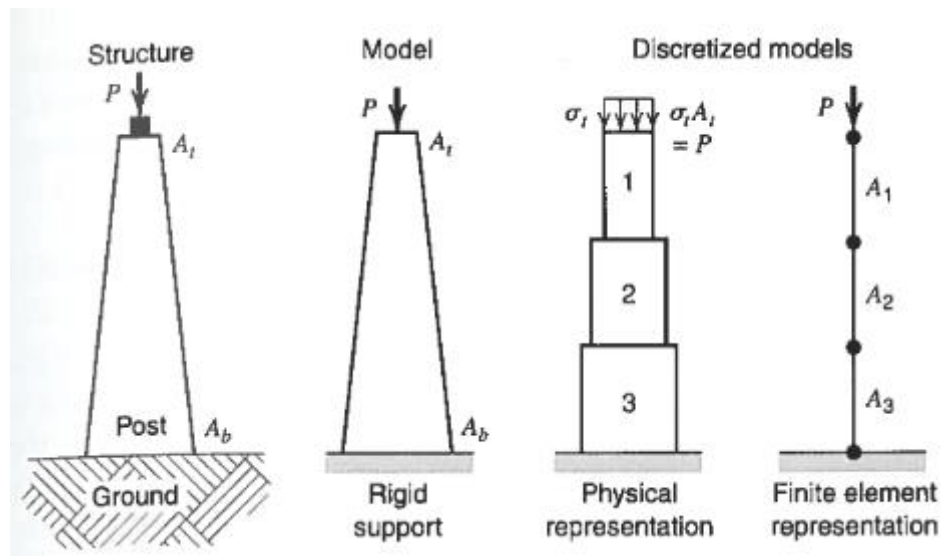


Figure 22. Steps from physical model to finite element model (Cook et al. 2001, 5)

In this case the only independent variable will be the axial coordinate, as well as only the magnitude of the loading matters, as assumption can be made that the stress is uniaxial at every cross section. Considering some simplifications described in section 4.1.1, at the end the finite element representation is quite simple yet accurate enough for this particular problem. In the model there are three elements connected together with total four nodes. To reduce the discretization error the number of elements can be increased. (Cook et al. 2001, 4-5.)



### 4.1.3 Numerical analysis

In this work the FEA process itself, which includes function interpolations and matrices calculations, will not be described very deeply as this is topic of a separate subject, but main ideas of the process will be discussed.

When using a FEA software, input data contains structure geometry, material properties, loading and boundary conditions. Meshing is done automatically, however, an engineer can redefine mesh partially if needed – increase the number of elements, mesh density, element type and so on. After this step the numerical analysis is performed.

The idea is that from each element's force matrix, displacement matrix and stiffness matrix the global matrices are calculated using system of equations and differential equations. From global matrices the final answers are got. All that is done automatically by software. The FEA solution and quantities are derived from the global matrix equation are displayed graphically or listed depending on what is required by an engineer. The only manual action in this step is to choose which results in what way should be displayed.

After that, the results should be examined qualitatively to make sure no rude errors appear, all deformations and stresses look adequately, all needed problems are solved, etc. (Cook et al. 2001, 13-14).

## 4.2 Results of finite element analysis

In this work ANSYS Workbench 18.0 software was used for modelling the problem and performing the finite element analysis.

The following four cases were chosen for comparing the results (dimensions explained in fig. 20):

Table 5. Relative dimensions for confirming the results

	Case I	Case II	Case III	Case IV
$R_1$	1	1	1	1
$R_2$	0.5	0.6	0.7	0.8
$e$	1.5	1.3	1.1	0.8

Fig. 23 – 26 are showing the results of finite method analysis with 100 MPa applied remote stress.

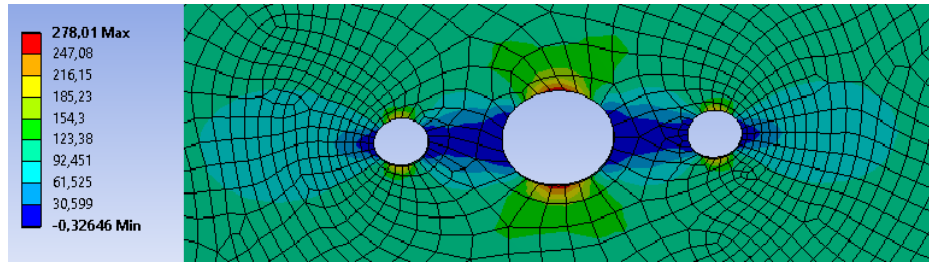


Figure 23. Case I FEA results

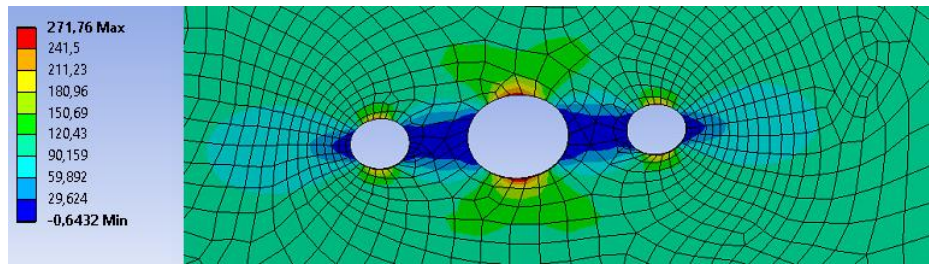


Figure 24. Case II FEA results

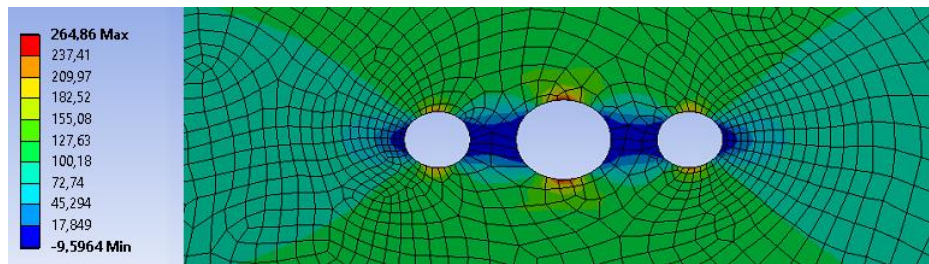


Figure 25. Case III FEA results

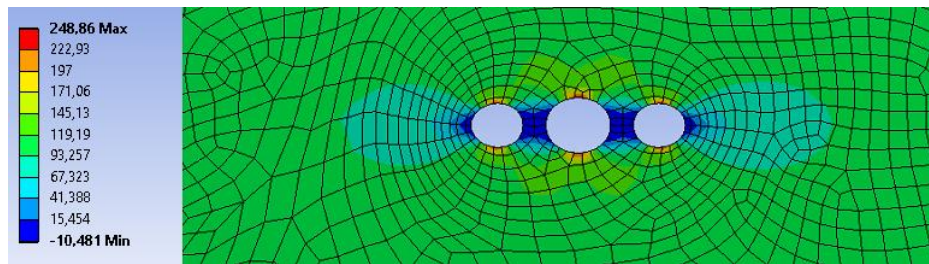


Figure 26. Case IV FEA results

Table 6 shows the maximum stress concentration factor's values of all cases.

Table 6. Maximum stress concentration factor by FEA

	Case I	Case II	Case III	Case IV
$K_t$	2.78	2.72	2.64	2.49

The results are compared and discussed in the last chapter of this work.

## 5 RESULTS AND DISCUSSION

Table 7 shows the stress concentration factors for cases I – IV obtained by different methods: body force method (Matlab code) and finite element method (ANSYS).

Table 7. Comparison of the results

	Case I	Case II	Case III	Case IV
BFM	2.81	2.72	2.61	2.46
FEA	2.78	2.72	2.64	2.49

As it comes from the results, Matlab code that used body force method shows pretty good correlation with finite element analysis as only 0.03 difference in the stress concentration values is the maximum (which is less than 1.2% error in all cases), so it can be concluded, that Matlab code can be used for estimating the stress concentration and optimizing the design.

As it is seen from the results (appendices 7-8), the stress concentration factor's reduction is the most noticeable when auxiliary holes are almost same size as the main hole and the distance between closest points of main hole and the auxiliary hole is about half of the radius of the main hole.

The following questions and tasks are the subjects of further studies of that topic:

- In 3D case there is a thumb rule: if between two holes the another one with the same diameter as smaller hole can be fitted in, then there will be no stress interaction, however, in 2D case the interaction is more severe. Does there exist such a thumb rule for 2D case and if yes, what is the criteria?
- To make a program, which uses Matlab code from this work, so no special software is needed to use the code.
- To confirm the most suitable results experimentally, using, for example, the tensile test.
- Make a handbook of stress intensities in case of two and three holes interaction, so it is easy to find smallest values.

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## MATLAB PROGRAM FOR A PLATE WITH ONE DRILLED HOLE

```

function [Kt] = One_hole(MM)
% Elliptical Hole in an Infinite Plate
v = 0 ;
epsilon = 0.000002;
a = 9 ;
b = 9 ;
t = zeros(MM,1);
sigmainf1 = ones(MM,1);
sigmainf2 = zeros(MM,1);
ellip = zeros(MM,2);
ellip_M = zeros(MM,1);
coord = zeros(MM,4);
coord_M = zeros(MM,2);
for N=1:MM
ellip(N,1) = -(pi/2)+2*(N-1)*pi/MM ;
ellip(N,2) = -(pi/2)+2*N*pi/MM;
coord(N,1) = a*cos(ellip(N,1));
coord(N,2) = b*sin(ellip(N,1));
coord(N,3) = a*cos(ellip(N,2));
coord(N,4) = b*sin(ellip(N,2));
ellip_M(N,1) = -(pi/2)+(2*N-1)*pi/MM ;
coord_M(N,1) = a*cos(ellip_M(N,1));
coord_M(N,2) = b*sin(ellip_M(N,1));
end
Sigmax_X = zeros(MM,MM);
Sigmax_Y = zeros(MM,MM);
Sigmay_X = zeros(MM,MM);
Sigmay_Y = zeros(MM,MM);
Tauxy_X = zeros(MM,MM);
Tauxy_Y = zeros(MM,MM);
for N=1:MM
for H=1:MM
coord_M(N,1) = a*cos(ellip_M(N,1));
coord_M(N,2) = b*sin(ellip_M(N,1));
x0 = @(q) a.*cos(q);
y0 = @(q) b.*sin(q);
x = coord_M(N,1);
y = coord_M(N,2);
l = @(q)(x-x0(q))./y;
m = @(q)(y-y0(q))./y;
F = @(q)1./(4.*pi.*y.*(l(q).^2+m(q).^2).^2);
Sigmax_X_sum = @(q) (-F(q).*l(q).*((3+v).*l(q).^2+(1-v).*m(q).^2)).*b.*cos(q);
Sigmax_Y_sum = @(q) (-F(q).*m(q).*((1+3.*v).*l(q).^2-(1-v).*m(q).^2)).*a.*sin(q);
Sigmay_X_sum = @(q) (F(q).*l(q).*((1-v).*l(q).^2-(1+3.*v).*m(q).^2)).*b.*cos(q);
Sigmay_Y_sum = @(q) (-F(q).*m(q).*((1-v).*l(q).^2+(3+v).*m(q).^2)).*a.*sin(q);

```

```

Tauxy_X_sum = @(q) (-F(q).*m(q).*((3+v).*l(q).^2+(1-v).*m(q).^2)).*b.*cos(q);
Tauxy_Y_sum = @(q) (-F(q).*l(q).*((1-v).*l(q).^2+(3+v).*m(q).^2)).*a.*sin(q);
if H ~ = N
Sigmax_X(N,H) = quad(Sigmax_X_sum,ellip(H,1),ellip(H,2));
Sigmax_Y(N,H) = quad(Sigmax_Y_sum,ellip(H,1),ellip(H,2));
Sigmay_X(N,H) = quad(Sigmay_X_sum,ellip(H,1),ellip(H,2));
Sigmay_Y(N,H) = quad(Sigmay_Y_sum,ellip(H,1),ellip(H,2));
Tauxy_X(N,H) = quad(Tauxy_X_sum,ellip(H,1),ellip(H,2));
Tauxy_Y(N,H) = quad(Tauxy_Y_sum,ellip(H,1),ellip(H,2));
elseif H == N
k=b/a ;
w=tan(ellip_M(H,1));
t(N,1) = atan((a/b)*tan(ellip_M(H,1)));
ADD_1 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(6*w^6+(3-2*k^2)*w^4-6*k^2*w^2-k^4) +
(-1/16)*(5+4*cos(2*t(N,1))-cos(4*t(N,1)));
ADD_2 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(-w^6+6*k^2*w^4+k^2*(6-k^2)*w^2-2*k^4) +
(1/16)*(1-cos(4*t(N,1)));
ADD_3 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(-2*w^6-(1-6*k^2)*w^4+6*k^2*w^2-k^4) +
(1/16)*(1-cos(4*t(N,1)));
ADD_4 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(-w^6-6*k^2*w^4+k^2*(2-k^2)*w^2+6*k^4) +
(-1/16)*(5-4*cos(2*t(N,1))-cos(4*t(N,1)));
ADD_5 = (epsilon*k^2/(4*pi*(k^2+w^2)^3))*(-9*w^5-(6+2*k^2)*w^3+k^2*(2-k^2)*w) +
(-1/16)*(2*sin(2*t(N,1))-sin(4*t(N,1)));
ADD_6 = (-epsilon*1/(4*pi*(k^2+w^2)^3))*((1-
2*k^2)*w^5+2*k^2*(1+3*k^2)*w^3+9*k^4*w) + (-
1/16)*(2*sin(2*t(N,1))+sin(4*t(N,1)));
Sigmax_X(N,H) = quad(Sigmax_X_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmax_X_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_1;
Sigmax_Y(N,H) = quad(Sigmax_Y_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmax_Y_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_2;
Sigmay_X(N,H) = quad(Sigmay_X_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmay_X_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_3;
Sigmay_Y(N,H) = quad(Sigmay_Y_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmay_Y_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_4;
Tauxy_X(N,H) = quad(Tauxy_X_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Tauxy_X_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_5;
Tauxy_Y(N,H) = quad(Tauxy_Y_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Tauxy_Y_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_6;
end
end
end
C_theta = zeros(MM,1);
S_theta = zeros(MM,1);
for N=1:MM
theta = atan((a/b)*tan(ellip_M(N,1)));
C_theta(N,1)= cos(theta);

```

```

S_theta(N,1)= sin(theta);
sigmainf1(N,1)= -1.*cos(theta);
end
Sigma_X1 = zeros(MM,MM);
Sigma_X2 = zeros(MM,MM);
Sigma_Y1 = zeros(MM,MM);
Sigma_Y2 = zeros(MM,MM);
for N = 1:MM
for H = 1:MM
Sigma_X1(N,H) = Sigmax_X(N,H)* C_theta(N,1)+Tauxy_X(N,H)*S_theta(N,1);
Sigma_X2(N,H) = Sigmax_Y(N,H)* C_theta(N,1)+Tauxy_Y(N,H)*S_theta(N,1);
Sigma_Y1(N,H) = Sigmay_X(N,H)* S_theta(N,1)+Tauxy_X(N,H)*C_theta(N,1);
Sigma_Y2(N,H) = Sigmay_Y(N,H)* S_theta(N,1)+Tauxy_Y(N,H)*C_theta(N,1);
end
end
sigma_inf = [sigmainf1;sigmainf2];
SIGMA1 = [Sigma_X1,Sigma_X2];
SIGMA2 = [Sigma_Y1,Sigma_Y2];
SIGMA = [SIGMA1;SIGMA2];
Kt = SIGMA \sigma_inf;
End

```

## MATLAB PROGRAM FOR A PLATE WITH TWO DRILLED HOLES

```

function[Kt] = Two_Holes(MM,s1,s2,a1,b1,a2,b2)
sigmainf2 = zeros(2*MM,1);
[ellip_M1,x1,y1] = ellip_MidPoint(s1,a1,b1,MM);
[ellip_M2,x2,y2] = ellip_MidPoint(s2,a2,b2,MM);
[Sigma_X1_1_1,Sigma_X2_1_1,Sigma_Y1_1_1,Sigma_Y2_1_1,sigmainf_1] =
MODULE(MM,ellip_M1,x1,y1,a1,b1,s1);
[Sigma_X1_1_2,Sigma_X2_1_2,Sigma_Y1_1_2,Sigma_Y2_1_2,sigmainf_1] =
MODULE_SP(MM,ellip_M1,x1,y1,a2,b2,a1,b1,s2);
Sigma_X1_1 = [Sigma_X1_1_1,Sigma_X1_1_2];
Sigma_X2_1 = [Sigma_X2_1_1,Sigma_X2_1_2];
Sigma_Y1_1 = [Sigma_Y1_1_1,Sigma_Y1_1_2];
Sigma_Y2_1 = [Sigma_Y2_1_1,Sigma_Y2_1_2];
[Sigma_X1_2_1,Sigma_X2_2_1,Sigma_Y1_2_1,Sigma_Y2_2_1,sigmainf_2] =
MODULE_SP(MM,ellip_M2,x2,y2,a1,b1,a2,b2,s1);
[Sigma_X1_2_2,Sigma_X2_2_2,Sigma_Y1_2_2,Sigma_Y2_2_2,sigmainf_2] =
MODULE(MM,ellip_M2,x2,y2,a2,b2,s2);
Sigma_X1_2 = [Sigma_X1_2_1,Sigma_X1_2_2];
Sigma_X2_2 = [Sigma_X2_2_1,Sigma_X2_2_2];
Sigma_Y1_2 = [Sigma_Y1_2_1,Sigma_Y1_2_2];
Sigma_Y2_2 = [Sigma_Y2_2_1,Sigma_Y2_2_2];
Sigma_X1 = [Sigma_X1_1;Sigma_X1_2];
Sigma_X2 = [Sigma_X2_1;Sigma_X2_2];
Sigma_Y1 = [Sigma_Y1_1;Sigma_Y1_2];
Sigma_Y2 = [Sigma_Y2_1;Sigma_Y2_2];
sigma_inf = [sigmainf_1;sigmainf_2;sigmainf2];
SIGMA1 = [Sigma_X1,Sigma_X2];
SIGMA2 = [Sigma_Y1,Sigma_Y2];
SIGMA = [SIGMA1;SIGMA2];
Kt = SIGMA \sigma_inf;
end

```



## MATLAB PROGRAM FOR A PLATE WITH THREE DRILLED HOLES

```

function[Kt] = Three_Holes(MM,s1,s2,s3,a1,b1,a2,b2,a3,b3)
sigmainf2 = zeros(3*MM,1);
[ellip_M1,x1,y1] = ellip_MidPoint(s1,a1,b1,MM);
[ellip_M2,x2,y2] = ellip_MidPoint(s2,a2,b2,MM);
[ellip_M3,x3,y3] = ellip_MidPoint(s3,a3,b3,MM);
[Sigma_X1_1_1,Sigma_X2_1_1,Sigma_Y1_1_1,Sigma_Y2_1_1,sigmainf_1] =
MODULE(MM,ellip_M1,x1,y1,a1,b1,s1);
[Sigma_X1_1_2,Sigma_X2_1_2,Sigma_Y1_1_2,Sigma_Y2_1_2,sigmainf_1] =
MODULE_SP(MM,ellip_M1,x1,y1,a2,b2,a1,b1,s2);
[Sigma_X1_1_3,Sigma_X2_1_3,Sigma_Y1_1_3,Sigma_Y2_1_3,sigmainf_1] =
MODULE_SP(MM,ellip_M1,x1,y1,a3,b3,a1,b1,s3);
Sigma_X1_1 = [Sigma_X1_1_1,Sigma_X1_1_2,Sigma_X1_1_3];
Sigma_X2_1 = [Sigma_X2_1_1,Sigma_X2_1_2,Sigma_X2_1_3];
Sigma_Y1_1 = [Sigma_Y1_1_1,Sigma_Y1_1_2,Sigma_Y1_1_3];
Sigma_Y2_1 = [Sigma_Y2_1_1,Sigma_Y2_1_2,Sigma_Y2_1_3];
[Sigma_X1_2_1,Sigma_X2_2_1,Sigma_Y1_2_1,Sigma_Y2_2_1,sigmainf_2] =
MODULE_SP(MM,ellip_M2,x2,y2,a1,b1,a2,b2,s1);
[Sigma_X1_2_2,Sigma_X2_2_2,Sigma_Y1_2_2,Sigma_Y2_2_2,sigmainf_2] =
MODULE(MM,ellip_M2,x2,y2,a2,b2,s2);
[Sigma_X1_2_3,Sigma_X2_2_3,Sigma_Y1_2_3,Sigma_Y2_2_3,sigmainf_2] =
MODULE_SP(MM,ellip_M2,x2,y2,a3,b3,a2,b2,s3);
Sigma_X1_2 = [Sigma_X1_2_1,Sigma_X1_2_2,Sigma_X1_2_3];
Sigma_X2_2 = [Sigma_X2_2_1,Sigma_X2_2_2,Sigma_X2_2_3];
Sigma_Y1_2 = [Sigma_Y1_2_1,Sigma_Y1_2_2,Sigma_Y1_2_3];
Sigma_Y2_2 = [Sigma_Y2_2_1,Sigma_Y2_2_2,Sigma_Y2_2_3];
[Sigma_X1_3_1,Sigma_X2_3_1,Sigma_Y1_3_1,Sigma_Y2_3_1,sigmainf_3] =
MODULE_SP(MM,ellip_M3,x3,y3,a1,b1,a3,b3,s1);
[Sigma_X1_3_2,Sigma_X2_3_2,Sigma_Y1_3_2,Sigma_Y2_3_2,sigmainf_3] =
MODULE_SP(MM,ellip_M3,x3,y3,a2,b2,a3,b3,s2);
[Sigma_X1_3_3,Sigma_X2_3_3,Sigma_Y1_3_3,Sigma_Y2_3_3,sigmainf_3] =
MODULE(MM,ellip_M3,x3,y3,a3,b3,s3);
Sigma_X1_3 = [Sigma_X1_3_1,Sigma_X1_3_2,Sigma_X1_3_3];
Sigma_X2_3 = [Sigma_X2_3_1,Sigma_X2_3_2,Sigma_X2_3_3];
Sigma_Y1_3 = [Sigma_Y1_3_1,Sigma_Y1_3_2,Sigma_Y1_3_3];
Sigma_Y2_3 = [Sigma_Y2_3_1,Sigma_Y2_3_2,Sigma_Y2_3_3];
Sigma_X1 = [Sigma_X1_1;Sigma_X1_2;Sigma_X1_3];
Sigma_X2 = [Sigma_X2_1;Sigma_X2_2;Sigma_X2_3];
Sigma_Y1 = [Sigma_Y1_1;Sigma_Y1_2;Sigma_Y1_3];
Sigma_Y2 = [Sigma_Y2_1;Sigma_Y2_2;Sigma_Y2_3];
sigma_inf = [sigmainf_1;sigmainf_2;sigmainf_3;sigmainf2];
SIGMA1 = [Sigma_X1,Sigma_X2];
SIGMA2 = [Sigma_Y1,Sigma_Y2];
SIGMA = [SIGMA1;SIGMA2];
Kt = pinv(SIGMA)*sigma_inf;

```

End

## MATLAB SUBPROGRAM "MODULE" FOR TWO AND THREE DRILLED HOLES

```

function [Sigma_X1,Sigma_X2,Sigma_Y1,Sigma_Y2,sigmaint1] =
MODULE(MM,ellip_M,x,y,a,b,s)
v = 0 ;
epsilon = 0.00002;
t = zeros(MM,1);
sigmoint1 = zeros(MM,1);
ellip = zeros(MM,2);
for N=1:MM
ellip(N,1) = -pi/2 + 2*(N-1)*pi/MM ;
ellip(N,2) = -pi/2 + 2*N*pi/MM;
end
Sigmax_X = zeros(MM,MM);
Sigmax_Y = zeros(MM,MM);
Sigmay_X = zeros(MM,MM);
Sigmay_Y = zeros(MM,MM);
Tauxy_X = zeros(MM,MM);
Tauxy_Y = zeros(MM,MM);
for N=1:MM
for H=1:MM
x0 = @(q) a.*cos(q);
y0 = @(q) b.*sin(q);
l = @(q)(x(N,1)-(s+x0(q)))./y(N,1);
m = @(q)(y(N,1)-y0(q))./y(N,1);
F = @(q)1./(4.*pi.*y(N,1).*(l(q).^2+m(q).^2).^2);
Sigmax_X_sum = @(q) (-F(q).*l(q).*((3+v).*l(q).^2+(1-v).*m(q).^2)).*b.*cos(q);
Sigmax_Y_sum = @(q) (-F(q).*m(q).*((1+3.*v).*l(q).^2-(1-v).*m(q).^2)).*a.*sin(q) ;
Sigmay_X_sum = @(q) (F(q).*l(q).*((1-v).*l(q).^2-(1+3.*v).*m(q).^2)).*b.*cos(q) ;
Sigmay_Y_sum = @(q) (-F(q).*m(q).*((1-v).*l(q).^2+(3+v).*m(q).^2)).*a.*sin(q) ;
Tauxy_X_sum = @(q) (-F(q).*m(q).*((3+v).*l(q).^2+(1-v).*m(q).^2)).*b.*cos(q) ;
Tauxy_Y_sum = @(q) (-F(q).*l(q).*((1-v).*l(q).^2+(3+v).*m(q).^2)).*a.*sin(q) ;
if H ~= N
Sigmax_X(N,H) = quad(Sigmax_X_sum,ellip(H,1),ellip(H,2));
Sigmax_Y(N,H) = quad(Sigmax_Y_sum,ellip(H,1),ellip(H,2));
Sigmay_X(N,H) = quad(Sigmay_X_sum,ellip(H,1),ellip(H,2));
Sigmay_Y(N,H) = quad(Sigmay_Y_sum,ellip(H,1),ellip(H,2));
Tauxy_X(N,H) = quad(Tauxy_X_sum,ellip(H,1),ellip(H,2));
Tauxy_Y(N,H) = quad(Tauxy_Y_sum,ellip(H,1),ellip(H,2));
elseif H == N
k=b/a ;
w=tan(ellip_M(N,1));
t(N,1) = atan((a/b)*tan(ellip_M(N,1)));
ADD_1 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(6*w^6+(3-2*k^2)*w^4-6*k^2*w^2-k^4) +
(-1/16)*(5+4*cos(2*t(N,1))-cos(4*t(N,1)));

```

```

ADD_2 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(-w^6+6*k^2*w^4+k^2*(6-k^2)*w^2-2*k^4)
+ (1/16)*(1-cos(4*t(N,1)));
ADD_3 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(-2*w^6-(1-6*k^2)*w^4+6*k^2*w^2-k^4) +
(1/16)*(1-cos(4*t(N,1)));
ADD_4 = (epsilon*k/(4*pi*(k^2+w^2)^3))*(-w^6-6*k^2*w^4+k^2*(2-k^2)*w^2+6*k^4)
+ (-1/16)*(5-4*cos(2*t(N,1))-cos(4*t(N,1)));
ADD_5 = (epsilon*k^2/(4*pi*(k^2+w^2)^3))*(-9*w^5-(6+2*k^2)*w^3+k^2*(2-k^2)*w)
+ (-1/16)*(2*sin(2*t(N,1))-sin(4*t(N,1)));
ADD_6 = (-epsilon*1/(4*pi*(k^2+w^2)^3))*((1-
2*k^2)*w^5+2*k^2*(1+3*k^2)*w^3+9*k^4*w) + (-
1/16)*(2*sin(2*t(N,1))+sin(4*t(N,1)));
Sigmax_X(N,H) = quad(Sigmax_X_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmax_X_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_1;
Sigmax_Y(N,H) = quad(Sigmax_Y_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmax_Y_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_2;
Sigmay_X(N,H) = quad(Sigmay_X_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmay_X_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_3;
Sigmay_Y(N,H) = quad(Sigmay_Y_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Sigmay_Y_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_4;
Tauxy_X(N,H) = quad(Tauxy_X_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Tauxy_X_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_5;
Tauxy_Y(N,H) = quad(Tauxy_Y_sum,ellip(H,1),ellip_M(H,1)-epsilon) +
quad(Tauxy_Y_sum,ellip_M(H,1)+epsilon,ellip(H,2)) + ADD_6;
end
end
end
C_theta = zeros(MM,1);
S_theta = zeros(MM,1);
for N=1:MM
theta = atan((a/b)*tan(ellip_M(N,1)));
C_theta(N,1)= cos(theta);
S_theta(N,1)= sin(theta);
sigmainf1(N,1)= -1.*cos(theta);
end
Sigma_X1 = zeros(MM,MM);
Sigma_X2 = zeros(MM,MM);
Sigma_Y1 = zeros(MM,MM);
Sigma_Y2 = zeros(MM,MM);
for N = 1:MM
for H = 1:MM
Sigma_X1(N,H) = Sigmax_X(N,H)* C_theta(N,1) + Tauxy_X(N,H)*S_theta(N,1);
Sigma_X2(N,H) = Sigmax_Y(N,H)* C_theta(N,1) + Tauxy_Y(N,H)*S_theta(N,1);
Sigma_Y1(N,H) = Sigmay_X(N,H)* S_theta(N,1) + Tauxy_X(N,H)*C_theta(N,1);
Sigma_Y2(N,H) = Sigmay_Y(N,H)* S_theta(N,1) + Tauxy_Y(N,H)*C_theta(N,1);
end
end
end

```

end

## MATLAB SUBPROGRAM "MODULE\_SP" FOR TWO AND THREE DRILLED HOLES

```

function [Sigma_X1,Sigma_X2,Sigma_Y1,Sigma_Y2,sigmaintf1] =
MODULE_SP(MM,ellip_M,x,y,a,b,a1,b1,s)
v = 0 ;
sigmainf1 = zeros(MM,1);
ellip = zeros(MM,2);
for N=1:MM
ellip(N,1) = -pi/2 + 2*(N-1)*pi/MM ;
ellip(N,2) = -pi/2 + 2*N*pi/MM;
end
Sigmax_X = zeros(MM,MM);
Sigmax_Y = zeros(MM,MM);
Sigmay_X = zeros(MM,MM);
Sigmay_Y = zeros(MM,MM);
Tauxy_X = zeros(MM,MM);
Tauxy_Y = zeros(MM,MM);
for N=1:MM
for H=1:MM
x0 = @(q) a.*cos(q);
y0 = @(q) b.*sin(q);
l = @(q)(x(N,1)-(s+x0(q)))./y(N,1);
m = @(q)(y(N,1)-y0(q))./y(N,1);
F = @(q)1./(4.*pi.*y(N,1).*(l(q).^2+m(q).^2).^2);
Sigmax_X_sum = @(q) (-F(q).*l(q).*((3+v).*l(q).^2+(1-v).*m(q).^2)).*b.*cos(q);
Sigmax_Y_sum = @(q) (-F(q).*m(q).*((1+3.*v).*l(q).^2-(1-v).*m(q).^2)).*a.*sin(q) ;
Sigmay_X_sum = @(q) (F(q).*l(q).*((1-v).*l(q).^2-(1+3.*v).*m(q).^2)).*b.*cos(q) ;
Sigmay_Y_sum = @(q) (-F(q).*m(q).*((1-v).*l(q).^2+(3+v).*m(q).^2)).*a.*sin(q) ;
Tauxy_X_sum = @(q) (-F(q).*m(q).*((3+v).*l(q).^2+(1-v).*m(q).^2)).*b.*cos(q) ;
Tauxy_Y_sum = @(q) (-F(q).*l(q).*((1-v).*l(q).^2+(3+v).*m(q).^2)).*a.*sin(q) ;
Sigmax_X(N,H) = quad(Sigmax_X_sum,ellip(H,1),ellip(H,2));
Sigmax_Y(N,H) = quad(Sigmax_Y_sum,ellip(H,1),ellip(H,2));
Sigmay_X(N,H) = quad(Sigmay_X_sum,ellip(H,1),ellip(H,2));
Sigmay_Y(N,H) = quad(Sigmay_Y_sum,ellip(H,1),ellip(H,2));
Tauxy_X(N,H) = quad(Tauxy_X_sum,ellip(H,1),ellip(H,2));
Tauxy_Y(N,H) = quad(Tauxy_Y_sum,ellip(H,1),ellip(H,2));
end
end
C_theta = zeros(MM,1);
S_theta = zeros(MM,1);
for N=1:MM
theta = atan((a1/b1)*tan(ellip_M(N,1)));
C_theta(N,1)= cos(theta);
S_theta(N,1)= sin(theta);
sigmainf1(N,1)= -1.*cos(theta);
end

```

```
Sigma_X1 = zeros(MM,MM);
Sigma_X2 = zeros(MM,MM);
Sigma_Y1 = zeros(MM,MM);
Sigma_Y2 = zeros(MM,MM);
for N = 1:MM
for H = 1:MM
Sigma_X1(N,H) = Sigmax_X(N,H)* C_theta(N,1) + Tauxy_X(N,H)*S_theta(N,1);
Sigma_X2(N,H) = Sigmax_Y(N,H)* C_theta(N,1) + Tauxy_Y(N,H)*S_theta(N,1);
Sigma_Y1(N,H) = Sigmay_X(N,H)* S_theta(N,1) + Tauxy_X(N,H)*C_theta(N,1);
Sigma_Y2(N,H) = Sigmay_Y(N,H)* S_theta(N,1) + Tauxy_Y(N,H)*C_theta(N,1);
end
end
end
```

## MATLAB SUBPROGRAM "ELLIP\_MIDPOINT" FOR TWO AND THREE DRILLED HOLES

```
function [ellip_M,x,y] = ellip_MidPoint(s,a,b,MM)
ellip = zeros(MM,2);
ellip_M = zeros(MM,1);
x = zeros(MM,1);
y = zeros(MM,1);
for N=1:MM
ellip(N,1) = -pi/2 + 2*(N-1)*pi/MM ;
ellip(N,2) = -pi/2 + 2*N*pi/MM;
ellip_M(N,1) = (ellip(N,1) + ellip(N,2))/2 ;
x(N,1) = s + a*cos(ellip_M(N,1));
y(N,1) = b*sin(ellip_M(N,1));
end
end
```



## PRELIMINARY RESULTS FOR A PLATE WITH TWO HOLES

$MM = 4$  value was used for these results.

$R_1$  is the radius of the main hole.

$R_2$  is the radius of the auxilliary hole.

$e$  is the distance between closer edges of the holes.

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.9918	3.0003	3.0045	3.0060	3.0063	3.0059	3.0052	3.0042	3.0032	3.0021	3.001

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.9979	3.0198	3.0259	3.0261	3.0238	3.0203	3.0161	3.0118	3.0076	3.0037	3.0003

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.0446	3.0621	3.0607	3.0532	3.0435	3.0334	3.0234	3.0141	3.0057	2.9984	2.9921

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.1094	3.1076	3.0912	3.0715	3.0521	3.0341	3.0179	3.0037	2.9914	2.9811	2.9725

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.1561	3.1302	3.0976	3.0663	3.0382	3.0139	2.9931	2.9755	2.9609	2.9489	2.9393

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.1554	3.1104	3.0667	3.0282	2.9956	2.9683	2.9458	2.9274	2.9125	2.9008	2.8917

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.0916	3.0394	2.9933	2.9545	2.9226	2.8967	2.8760	2.8596	2.8469	2.8374	2.8304

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.9640	2.9184	2.8797	2.8477	2.8220	2.8017	2.7860	2.7743	2.7660	2.7603	2.7570

## Appendix 7/2

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.7835	2.7569	2.7337	2.7144	2.6992	2.6878	2.6798	2.6749	2.6724	2.6721	2.6736

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.5682	2.5684	2.5658	2.5628	2.5608	2.5605	2.5619	2.5650	2.5696	2.5755	2.5824

## PRELIMINARY RESULTS FOR A PLATE WITH THREE HOLES

$MM = 4$  value was used for these results.

$R_1$  is the radius of the main hole.

$R_2 = R_3$  is the radius of the auxilliary holes.

$e$  is the distance between closer edges of the holes.

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.9808	2.9990	3.0056	3.0078	3.0080	3.0073	3.0059	3.0042	3.0024	3.0007	2.9992

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.9945	3.0261	3.0340	3.0330	3.0284	3.0221	3.0150	3.0080	3.0013	2.9952	2.9900

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.0648	3.0830	3.0757	3.0609	3.0442	3.0273	3.0113	2.9968	2.9839	2.9728	2.9635

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.1525	3.1327	3.0988	3.0642	3.0325	3.0042	2.9795	2.9582	2.9403	2.9256	2.9137

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.1905	3.1288	3.0694	3.0182	2.9752	2.9392	2.9094	2.8849	2.8651	2.8495	2.8374

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.5	0.5	0.5	0.5	0.5	0.5
$e$	1.1	1.2	1.3	1.4	1.5	1.6
$K_t$	2.8284	2.8220	2.8177	2.8152	2.8143	2.8145

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$e$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	3.1260	3.0392	2.9676	2.9101	2.8641	2.8273	2.7981	2.7751	2.7575	2.7444	2.7350

$R_1$	1.0	1.0	1.0	1.0
$R_2$	0.6	0.6	0.6	0.6
$e$	1.1	1.2	1.3	1.4
$K_t$	2.7288	2.7252	2.7238	2.7242

## Appendix 8/2

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.9428	2.8578	2.7918	2.7409	2.7015	2.6714	2.6487	2.6320	2.6204	2.6130	2.6091

$R_1$	1.0	1.0
$R_2$	0.7	0.7
e	1.1	1.2
$K_t$	2.6080	2.6092

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.6606	2.6011	2.5561	2.5224	2.4977	2.4803	2.4688	2.4622	2.4597	2.4605	2.4640

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.3188	2.2978	2.2934	2.3140	2.3233	2.3496	2.3666	2.3835	2.4003	2.4171	2.4339

$R_1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$R_2$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
e	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K_t$	2.4104	2.4215	2.4298	2.4370	2.4442	2.4520	2.4604	2.4696	2.4795	2.4899	2.5007