CALCULATION OF BENDING AND SHEAR MODULUS OF THIN PLATE BY VIBRATION ANALYSIS

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Abstract:

The thesis aims to verify the possibility of applying vibrational measuring method on thin plate made of light-weighed material. Particularly, it aims to develop formula for the calculation of dynamic shear and bending modulus for light-weighed material with such method. This method has been proven and being applied on heavy-weighed material such as metals and alloy based on standard ASTM E1876-15 which is explained in the Literature review part.

In ASTM E1876 method, the vibration of the specimen is caused by knocking a hammer on a metal plate placed upon two supports. However, this experiment is not possible to produce with light weighed materials as they will fly off the support. Therefore, the possibility of performing vibrational measuring method on light weighed materials needs to be testified.

In order for the method to be proven reliable, it needs to be supported by both theory, simulation and practical experiments. The theory is developed based on already existed theory of the same method on high density material; and the simulation is complemented with COMSOL multiphysics. The practical experiments are planned and done according to the availability of the equipment at Arcada's laboratory. There is a small part of experiment remained undone due to technical issue. However, detailed set up of the apparatus for the experiment is well provided along with some alternatives in case failure occurs.

After studying the matter at many particular cases and different points of view, it is concluded that the method is applicable for the purposed material. In other words, it provides an alternative for standard ASTM E 1876-15 suitable for low density material.

Cantilever beam vibration mathematics is tested using Comsol and with 3-point bending and Cantilever beam vibration. The results show that the error of Young's modulus calculation between Cantilever beam vibration and 3 points bending is 1,6%. The Young's modulus results compared between experiment of the 3-point bending at Arcada and the 3-point bending at Borås is 1,694%

The study of the thesis proves that Cantilever beam vibration has a known solution and easily be done on light weighed material when the specimen is clamped properly.

COMSOL software used as Finite Element Method helps not only to verify theoretical formulas, but also to develop a formula for calculating the frequency of the torsional vibration of a thin beam hinged at both ends

$$f_t = 0{,}5\sqrt{\frac{JG}{LI_L}}$$

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1 INTRODUCTION

1.1 Objectives

The primary purpose of the thesis is to determine the mechanical properties of material based on the calculation of material dynamic shear and bending modulus; from which Poisson number would be found.

The reviews of existing standards and methods used to measure the shear and bending modulus of heavy-weighed materials by vibrational testing method will be mentioned, followed by the examination in the case of light-weighed material.

The thesis aims to find out all the alternatives for calculating shear and dynamic Young's modulus for low density material with vibrational method by relating with the theory of the case of heavy-weighed material and using Finite Element Method.

The most important objects of the thesis can be summed up in two questions:

1. Is the mentioned dynamic testing applicable for light-weighed materials?
2. What are the mathematical formulas of shear and bending modulus that should be recommended for light-weighed materials, in case the current existing standard does not provide a possible method?

1.2 The relevance of the problem

Inspecting mechanical properties of materials is an important part of the manufacturing process. Throughout the development of the technology industry, mechanical testing has been developed and improved itself to achieve a wide range of testing methods with reliable results. These can be categorized into two main groups: Destructive method and Non-destructive method (NDM).

Destructive testing methods are widely used to determine the physical properties of materials such as durability, ductility, hardness, stiffness, surface roughness, Young modulus, yield and ultimate tensile strength. This way of finding mechanical properties of materials is simple, effective and reliable, however it causes damages or even completely breaks the samples or parts. During the real-life inspection work, one would not preferably deform or break a testing part, but instead, remain its state, and still achieve the purpose of examining the mechanical properties of the parts. Non-destructive testing method stands out to solve the problem.

Non-destructive testing is the process of testing and inspecting of materials that causes no physical damage to the testing object and remains it in its current state. In other words, the part is in normal operation after the test. Non-destructive testing has been applied to various fields of industry such as aerospace, automobiles, pipeline and shipping industry. In shipping industry, for example, Non-destructive testing is an unreplaceable part of the maintenance procedure in which parts are inspected to collect necessary information that would help predict the working capability of the parts and give the maintenance plan.

The techniques that are commonly used in NDM are visual inspection, magnetic particle testing, ultrasonic testing and liquid penetrant testing. In this thesis, the focus will be on other technique which is called Dynamics testing. This method has been successfully exploited by a company named IMCE which will be referred to in the next part of this document.

Dynamic testing is a type of non-destructive mechanical testing that are done by standard. It relies on the recording of vibrational signal produced when a sample is tapped with a small projectile. This method has shown reliable results and been used extensively in IMCE. The results are then analyzed and used in order to figure out the aging of material from which long maintenance or repair plans can be scheduled. Other important information about material is also found out such as current Shear Modulus, Young's modulus, Poisson's ratio.

1.3 Relationship to existing knowledge

The theory is based partly on the existing standard applied to calculated shear and dynamic Young’s modulus, ASTM E1876-15, which will be discussed in the later section.

It is necessary to refer to IMCE. This is a company that is currently using standard ASTM E1876-15 in their business. The company is major in material analysis and related fields. More information about IMCE is found in the link given in section 7: REFERENCES

2 LITERATURE REVIEW

2.1 ASTM E1876-15

ASTM E1876 refers to a standard method for calculating dynamic Young's modulus, shear modulus, and Poisson's ratio by Impulse excitation of Vibration. Beside ASTM E1876-15 standard, it could be seen simultaneously standards ISO 12680-1 and EN 843-2 which are not described in detail but briefly as below:

1. ISO 12680-1 refers to methods of test for refractory products - part 1: Determination of dynamic Young's modulus (MOE) by impulse excitation of vibration.
2. EN 843-2 refers to advanced technical ceramics – Mechanical properties of monolithic ceramics at room temperature – part 2: Determination of Young's modulus, shear modulus and Poisson's ratio.

2.1.1 Impulse excitation technique

Impulse excitation technique is a non-destructive testing method used to measure elastic properties and internal friction of materials. By observing the resonant frequencies, it can give measurement of Young's modulus, shear modulus, Poisson's ratio and internal friction of predefined shapes like rectangular bars, cylindrical rods and disc shaped samples.

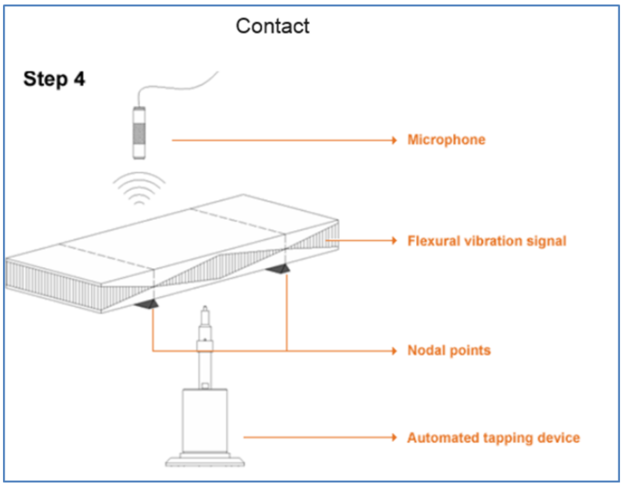

Figure 1: Impulse excitation technique [1]

An automated tapping device is used to produce regular tapping at the tested material with a small projectile. The induced vibrational signal is then recorded with a microphone. Afterward, the acquired vibrational signal in the time domain is converted to the frequency domain by a fast Fourier transformation. Dedicated software will determine resonant frequency to calculate elastic properties based on the beam theory. [1]

2.1.2 Elastic properties

1. Dynamic Young's modulus

$$E = 0{,}9465\left(\frac{m f_f^2}{w}\right)\left(\frac{L^3}{t^3}\right) R_1 \quad [1] \tag{1}$$

With $R_1 = 1 + 6{,}585\left(\frac{t}{L}\right)^2$, and $\frac{L}{t} \geq 20$ [2]

Table 1: Terms and definition

Term	Definition	Unit		Term	Definition	Unit
E	Young's modulus	kg m^{-1} s^{-2}		w	Width	m
f_f	Flexural frequency	s^{-1}		t	Thickness	m
m	Mass	kg		R_1	Correction factor	Dimensionless
L	Length	m				

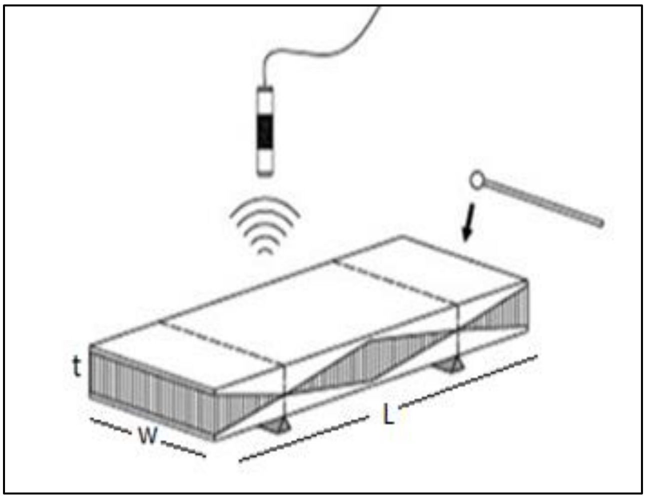

Figure 2: Young's modulus calculation [1]

Notice that when $\frac{L}{t} \geq 20$ then R_1 is close to 1. The table below gives some values of R_1 according to the value of *L* and *t*.

Table 2: Correction factors

L(m) Length	**t(m) Thickness**	**w(m) Width**
0,1	0,005	0,0100
0,11	0,005	0,0110
0,12	0,005	0,0120
0,13	0,005	0,0130
0,14	0,005	0,0140
0,15	0,005	0,0150
0,16	0,005	0,0160
0,17	0,005	0,0170
0,18	0,005	0,0180
0,19	0,005	0,0190

Young’s modulus is calculated through the measurement of flexural vibration frequency and the mass and dimensions of sample according to the different standards (ASTM E1876-15, ISO 12680-1, EN 843-2). [1]

2. Shear Modulus

$$G = \frac{4Lmf_t^2}{wt} R_2 \ [1] \qquad (2)$$

With

$$R_2 = \left(\frac{1+\left(\frac{w}{t}\right)^2}{4-2{,}521\frac{t}{w}\left(1-\frac{1{,}991}{e^{\frac{w}{t}}+1}\right)} \right) \left(1 + \frac{0{,}00851 w^2}{L^2}\right) - 0{,}060 \left(\frac{w}{L}\right)^{\frac{3}{2}} \left(\frac{w}{t} - 1\right)^2 \ [2]$$

Assume that $w > t$.

Table 3: Terms and definition

Term	Definition	Unit		Term	Definition	Unit
E	Young's modulus	kg m^{-1} s^{-2}		w	Width	m
f_t	Torsional frequency	s^{-1}		t	Thickness	m
m	Mass	kg		R_2	Correction factor	Dimensionless
L	Length	m				

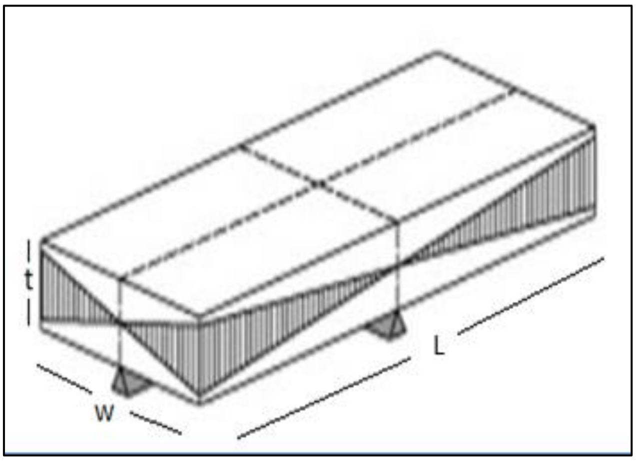

Figure 3: Shear modulus calculation [1]

The table below gives some values of R_2 according to the value of *w, t, L*

Table 4: Correction factor

L(m) **Length**	**t(m)** **Thickness**	**w(m)** **Width**	**R_2** **Correction factor**
0,1	0,005	0,0100	1,743788
0,1	0,005	0,0110	1,978668
0,1	0,005	0,0120	2,234256
0,1	0,005	0,0130	2,510083
0,1	0,005	0,0140	2,805802
0,1	0,005	0,0150	3,121148
0,1	0,005	0,0160	3,455908
0,1	0,005	0,0170	3,809897
0,1	0,005	0,0180	4,182953
0,1	0,005	0,0190	4,574926

To calculate the shear modulus, the equipment measures the torsional vibration frequency and determine the shear modulus using the mass and dimensions of sample according to the standards (ASTM E1876-15, ISO 12680-1, EN 843-2). [1]

3. Poisson’s ratio

Poisson's ratio is determined by using Hooke's law which can only be applied to isotropic materials (ASTM E1876-15, ISO 12680-1, EN 843-2). [1]

$$\upsilon = \frac{E}{2G} - 1 \ \ [1] \tag{3}$$

With υ being the Poisson's ratio

E is the value of the Young's modulus

G is the value of the shear modulus

2.2 Equation of damped harmonic motion

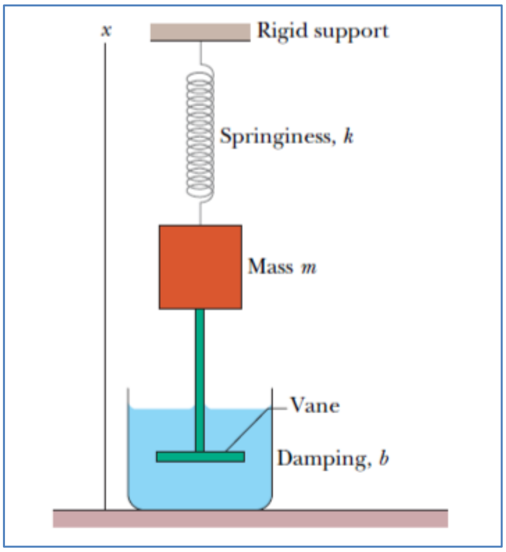

Figure 4: Damped harmonic motion [3]

When there is an external force exert on an oscillator and cause the motion die out over time, it is said that the motion is damped. In the figure above, the damping force is the dragging force of the water exerting on the vane.

In a simple case, assume that the damping force $\overrightarrow{F_d}$ is proportional to the velocity v, then.

$$F_d = -bv \quad [3] \tag{4}$$

Where b is a damping constant that depends on the environmental factors.

The force on the block from the spring is $F_s = -kx$; and $F_{net} = ma$ where F_{net} is the net force acting on the block. Then we have:

$$-bv - kx = ma \tag{5}$$

$$I.e.., m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{6}$$

The solution for this is:

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega_d t + \theta) \tag{7}$$

Where x_m is the amplitude and ω_d is the angular frquency of the damped oscillator. This angular frequency is given by:

$$\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{8}$$

If $b = 0$ there is no damping.

If b is small but not zero, then ω_d is approximate the freqency of undamped oscillator, $\omega_N = \sqrt{\frac{k}{m}}$; then remind that the mechanical energy of a simple harmonic motion without damping is given by $\epsilon = \frac{1}{2}k{x_m}^2$ [4], where ϵ is the mechanical energy, k is the spring stiffness and x_m is the amplitude of the harmonic motion; by replacing the amplitude from equation (7) we get:

$$\epsilon(t) \approx \frac{1}{2} x_m^2 e^{\frac{-bt}{m}} \text{ [3]} \tag{9}$$

Which tells that in damped harmonic motion, like the amplitude, the mechancal energy decreases exponentially over time.

It is seen that in air-vacuumed condition, which implies $b = 0$, the freqency of the motion is $\omega_d = \omega_N = \sqrt{\frac{k}{m}}$ which is only dependent on k – the spring's stiffness:

$$\omega_N = \sqrt{\frac{k}{m}} \tag{10}$$

2.3 Torsion of a rectangular massless beam

Consider a rectangular uniformed-section beam fixed at one end and free to move at the other end. The beam is subject to a torque of magnitude T. Assume that the mass of the beam is relatively small compared to the magnitude of the torque.

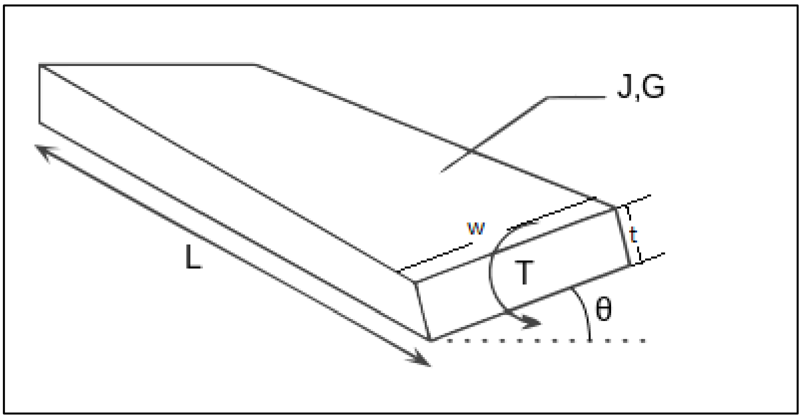

Figure 5: Torsion of massless beam [5]

An important characteristic of torsional deformation of rectangular bar is the warping of the cross section. On a rectangular cross section, the maximum shear stress (τ_{max}) takes place at the middle of the longer edge. The maximum shear stress of a rectangular beam is given by:

$$\tau_{max} = \frac{T}{\alpha \times wt^2} \quad [9] \tag{11}$$

Where T is the applied torque, w and t be are the dimensions of the bar as seen in the above figure; α is a dimensionless constant obtained by a theory of elasticity solution and listed in the table below.

The angle of twist θ is given by:

$$\theta = \frac{TL}{JG} \quad [9] \tag{12}$$

Where $J = \beta wt^3$ is the torsional constant; G is the modulus of rigidity (shear modulus) of the material; and β is a constant listed in the table below.

Table 5: Torsion constant for rectangular bars [6]

$\boldsymbol{\frac{w}{t}}$	1,00	1,50	1,75	2,0	2,50	3,00	4	6	8	10	∞
$\boldsymbol{\alpha}$	0,208	0,231	0,239	0,246	0,258	0,267	0,282	0,298	0,307	0,312	0,333
$\boldsymbol{\beta}$	0,141	0,196	0,214	0,229	0,249	0,263	0,281	0,298	0,307	0,312	0,333

3 METHOD

In this part, the theory about the vibration of a beam in some cases will be developed and explained.

3.1 The deflection of a Cantilever beam with external mass

Consider a cantilever beam fixed to a vertical support at one end and subject to a load at the other end; and assume that the mass of the beam is relatively small compared to the external mass.

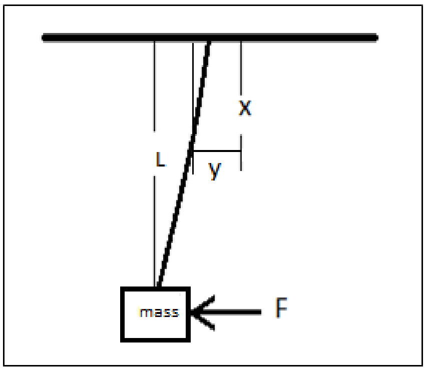

Figure 6: Cantilever beam structure [7]

The deflection y at any point x along the beam is calculated by:

$$y = \frac{Fx^2}{6EI_A}(3L - x) \quad [7] \tag{13}$$

The deflection y_{max} at the free end of the beam is then given by:

$$y_{max} = \frac{FL^3}{3EI_A} \tag{14}$$

Table 6: Terms and definition

Term	Definition	Unit
E	Modulus of elasticity (Young's modulus)	kg m^{-1} s^{-2}
I_A	Area moment of inertia	m^4
F	Load	kg m s^{-2}
L	Projectile length of the beam unto the ground	m
x	Distance from the section to the support	m
y	deflection	m

According to Hooke's law we have:

$$F = kx$$

Where F is the applied force, k is the spring's stiffness and x is small compared to the spring's length.

From equation (14) we have:

$$x = \frac{F}{k} \tag{15}$$

Now combine equation (14) and (15) with the notice that in equation (11), y_{max} plays the same role as x in equation (15), we get:

$$k = \frac{F}{y} = \frac{F}{\frac{FL^3}{3EI_A}} = \frac{3EI_A}{L^3} \tag{16}$$

From equation (10) and (16) we then have:

$$\omega_N = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI_A}{mL^3}} \tag{17}$$

Notice that equation (17) is true only when no damping occurs.

Note that the formula for area moment of inertia I_A for a bar of width *w* and thickness *t* is:

$$I_A = \frac{1}{12}wt^3 \tag{18}$$

From equation (17) and (18) we have:

$$\omega_N = \sqrt{\frac{k}{m}} = \sqrt{\frac{Ewt^3}{4mL^3}} \tag{19}$$

The frequency in Hz is $f_N = \frac{\omega_n}{2\pi}$, combine with equation (19) we get:

$$E = 16\pi^2 {f_N}^2 \left(\frac{m}{w}\right) \left(\frac{L}{t}\right)^3 \tag{20}$$

Remind that in equation (1) we have $E = 0{,}9465\left(\frac{mf_f^2}{w}\right)\left(\frac{L^3}{t^3}\right)R_1$, which is similar to equation (20). Notice that the only difference between the two equations is the constant factors: in equation (20), it is $16\pi^2$ and in equation (1), $0{,}9465R_1$. This similarity shows that the boundary conditions change, but the system remains constant.

3.2 Free vibration of a Cantilever beam with distributed mass.

Consider a cantilever beam, with length L, clamped at one end and subjected to free vibration at the other end. The mass of the beam is considered to be evenly distributed along its length. We aim to find the natural frequency of the beam under vibration.

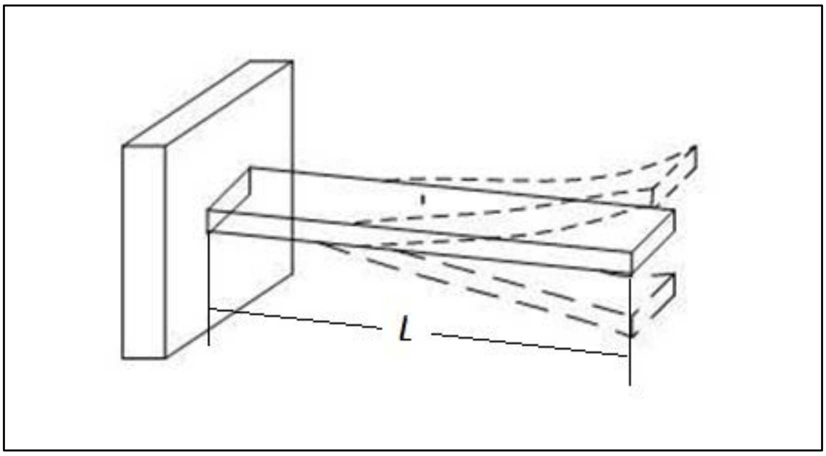

Figure 7: Cantilever beam subjected to free vibration [8]

According to Euler-Bernoulli Beam Theory, we find:

$$EI_A \frac{d^4y}{dx^4} + \rho A \frac{d^2y}{dt^2} = 0 \text{ [9]} \tag{21}$$

Where, E is the modulus of rigidity of beam material, I_A is the area moment of the beam cross-section, A is the cross-section area, ρ is the material density, x and y are the displacement from the fixed end in x and y axis at time t respectively.

The normal solution for the above equation is:

$$Y(x,t) = X(x)T(t) \tag{22}$$

From equation (21) and (22) we have:

$$\frac{EI_A}{\rho A}\frac{d^4X}{dx^4}T(x)-X(t)\frac{d^2T}{dt^2}=0 \tag{23}$$

Solution for displacement is given by:

$$X(x)=C_1Cos(x\delta)+C_2Sin(x\delta)+C_3Cosh(x\delta)+C_4Cosh(x\delta)\ \ [9] \tag{24}$$

In which,

$$\delta=\left(\frac{\rho A}{EI_A}\omega^2\right)^{\frac{1}{4}} \tag{25}$$

Because one end is fixed, therefore the displacement and slope are zero at this end which implies:

$$At\ x=0, y=0, \frac{dy}{dx}=0$$

While at the free end, the moment is zero $\left(\frac{d^2y}{dx^2}=0\right)$ and shear is zero $\left(\frac{d^3y}{dx^3}=0\right)$. Thus, we have the boundary condition:

$$At\ x=0, y=0, \frac{dy}{dx}=0$$

$$At\ x=L, \frac{d^2y}{dx^2}=0, \frac{d^3y}{dx^3}=0$$

Apply above conditions to equation (24), we get $C_2=-C_4$ and $C_1=-C_3$. Solving this for C_1 and C_2 we find:

$$Cos(\delta l)Cosh(\delta l)=-1 \tag{26}$$

The general solution of equation (26) is $\delta_nL=\frac{(2n-1)\pi}{2}$. Combining this with equations (24) and (25) gives:

$$T(t) = b_1 Sin\left[\left(\delta_n{}^2\sqrt{\frac{EI_A}{\rho A}}\right)t\right] + b_2 Cos\left[\left(\delta_n{}^2\sqrt{\frac{EI_A}{\rho A}}\right)t\right] \quad (27)$$

The frequency in rad/s is:

$$\omega_n = \frac{\delta_n{}^2}{L^2}\sqrt{\frac{EI_A}{\rho A}} \quad [9] \quad (28)$$

The natural frequency in Hz is:

$$f_n = \frac{\omega_n}{2\pi} = \frac{\delta_n{}^2}{2\pi L^2}\sqrt{\frac{EI_A}{\rho A}} \quad (29)$$

The shapes of the first three modes are shown as below:

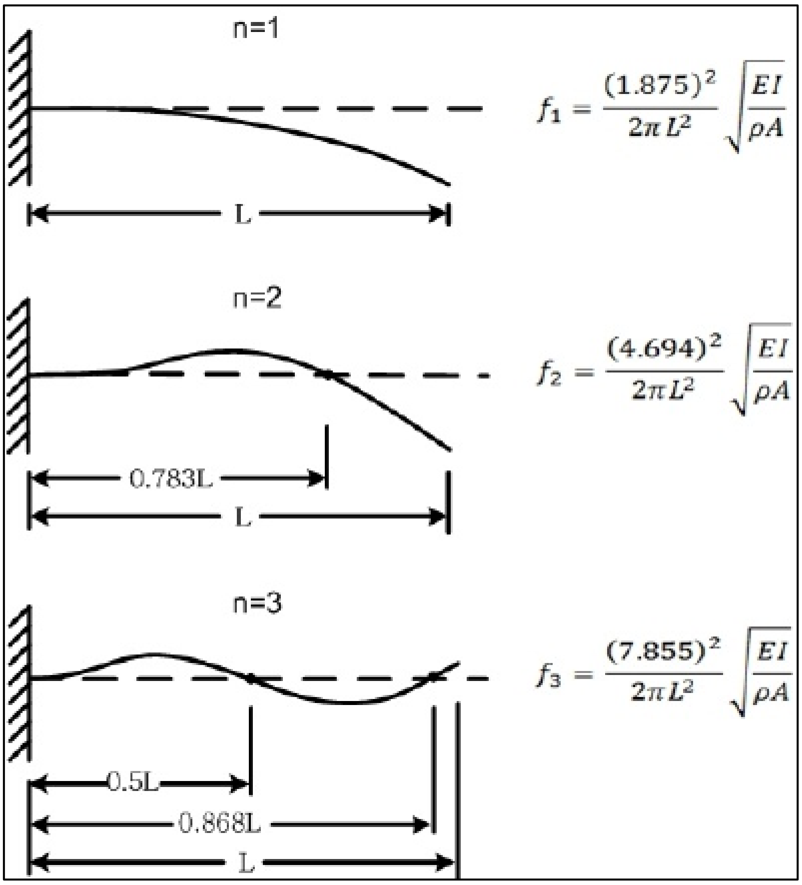

Figure 8: Mode shapes and natural frequencies for the first three modes of flexural vibration of the cantilever beam [10]

Notice that $\frac{f_1}{f_2} \approx 0{,}1596$. In the next part, we will recheck this ratio with COMSOL software to prove that the software gives the results that match the theory.

In equation (29), solve for E and remind that, $\rho = \frac{mass}{volume} = \frac{m}{L.A}$, we get, for the first mode:

$$E = {f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right) \left(\frac{48\pi^2}{1{,}875^4}\right) \qquad (30)$$

Again, compare equation (30) to equation (1), $E = 0{,}9465 \left(\frac{mf_f^2}{w}\right)\left(\frac{L^3}{t^3}\right) R_1$ we can recognize the similarity between them. The only difference between them is the boundary condition resulting in the factor $\frac{48\pi^2}{1{,}875^4} = 38{,}33$ in equation (30) and $0.9465R_1$ in equation (1). This shows gain that the boundary condition does not affect the system but only the constant factor.

3.3 Torsional harmonic oscillator

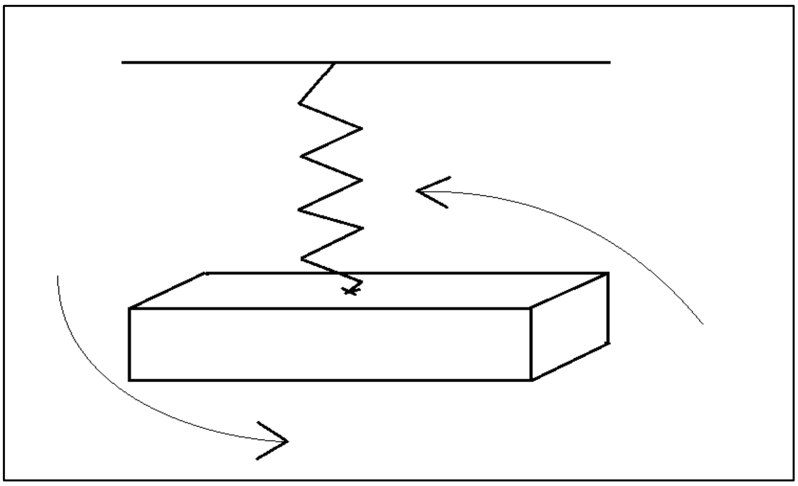

Figure 9: Torsional Pendulum [11]

Suspend a bar from a thin wire and wind it by an angle θ, the bar swings and due to the restoring force of the wire caused by torsion. The motion of the bar is an example of the motion a torsional harmonic oscillator. The torsional torque is (the term definitions are given the table below):

$$\mathrm{T} = -\kappa\theta \text{ [11]} \tag{31}$$

The equation of the motion is:

$$I\theta'' = -\kappa\theta \tag{32}$$

We can rewrite equation (32) as:

$$I\frac{d^2\theta}{dt^2} + C\frac{d\theta}{dt} + \kappa\theta = T(t) \quad \text{[11]} \tag{33}$$

If the damping is small, $C \ll \sqrt{\frac{\kappa}{I}}$, the frequency of the vibration is close to the natural resonant frequency of the system:

$$\omega_N = \sqrt{\frac{\kappa}{I}} \tag{34}$$

The general solution for equation (33) in case there is no driving force which mean $\mathrm{T}(t) = 0$ is:

$$\theta = Ae^{-\alpha t} cos(\omega t + \phi) \quad [11] \tag{35}$$

Where $\varrho = \frac{C}{2I}$ and $\omega_d = \sqrt{{\omega_N}^2 - \varrho^2} = \sqrt{\frac{\kappa}{I} - \left(\frac{C}{2I}\right)^2}$.

Table 7: Terms and definition

Term	**Unit**	**Definition**
θ	rad	Twisted angel
I	kg m^2	Moment of inertia of the bar
C	J s rad^{-1}	Angular damping constant
κ	N m rad^{-1}	Torsion spring constant
T	N m	Drive torque
ω_N	rad s^{-1}	Un-damped resonant frequency in radian
ω_d	rad s^{-1}	Damped resonant frequency in radian
ϱ	s^{-1}	Reciprocal of damping time constant
ϕ	rad	Phase angle of oscillation

Consider the case when we replace the wire by a thin bar of length L, width w and thickness t ($w \gg t$) and of negligible mass. The bar is now replaced by a shaft of cylinder shape with length l and mass m.

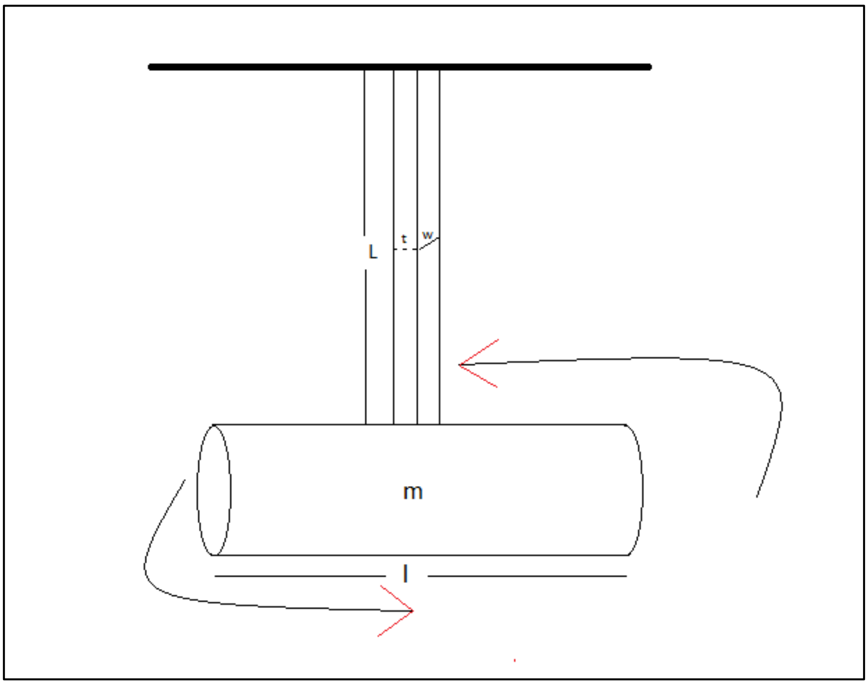

Figure 10: twisting motion

Equation (12) gives that the angle of twist is $\theta = \frac{TL}{JG}$ which implies that:

$$T = \theta \frac{JG}{L} \tag{36}$$

Equation (31) gives that $\mathrm{T} = -\kappa\theta$. The negative sign is taken away when we consider the magnitude of the torque only. Then we have:

$$\theta \frac{JG}{L} = \kappa\theta \tag{37}$$

Thus,

$$\kappa = \frac{JG}{L} \tag{38}$$

Equation (34) gives $\omega_N = \sqrt{\frac{\kappa}{I}}$ .Substitute κ from equation (38) we get:

$$\omega_N = \sqrt{\frac{JG}{IL}} \tag{39}$$

Remind that I is the moment of inertia of the cylinder shaft with the axis of rotation, in this case, is at the center of the shaft. We have:

$$I = \frac{1}{12} ml^2 \tag{40}$$

Where m and l are the mass and length of the shaft as defined above.

Substitute I from equation (39) to equation (40) and notice that $J = \beta wt^3 = \frac{1}{3}wt^3$ as $w \gg t$ (check Table 1), we get.

$$\omega_N = \sqrt{\frac{12 \times JG}{Lml^2}} = \sqrt{\frac{4wt^3G}{Lml^2}} \tag{41}$$

We expect that the torsional frequency of twist is proportional to $\omega_N \propto \frac{1}{\sqrt{L}}$. This result is valid only if there is no or insignificant damping. As we discussed earlier the damping is related to the frequency of the vibration such that the damping is big if the frequency is big. Therefore equation (41) is possible if ω_N is small. This occurs when L is large compared to w and t.

Solve equation (41) for G and notice that $\omega_N = 2\pi f_N$, we get:

$$G = \frac{\pi^2 {f_N}^2 Lml^2}{wt^3} = \frac{Lm{f_N}^2}{wt} \times \frac{\pi^2 l^2}{t^2} \tag{42}$$

Equation (42) gives a theoretical formula to calculate the Shear modulus of the thin rectangular bar by mean of the frequency ω_N.

Now if we look back at equation (2) $G = \frac{Lmf_t^2}{wt} \times 4R_2$ we can see that the difference between the two equations is the two factors $\frac{\pi^2 l^2}{t^2}$ in equation (42) and $4R_2$ in equation (2).

$$G = \frac{4Lm{f_t}^2}{wt} R_2$$

3.4 COMSOL simulation

COMSOL Multiphysics software is a simulation software for modelling designs, devices and processes in the field of engineering technology and research. The software provides designing tools that help model and simulate variety of mechanical test.

In the project, the version 5.3a of COMSOL Multiphysics – Classkit license is used to perform simulations as part of theoretical research.

In this section, some of the theories that are mentioned earlier are modelled and sent to simulation with the software. In other words, we recheck the formulas that have been developed earlier with COMSOL Multiphysics.

The purpose of this task is in the first place, to re-enforce the theory and in the second place, to prove the reliability of COMSOL Multiphysics software.

3.4.1 Eigen frequency ratios and Young's modulus

In the very end of the section 3.2, we come up with the two important results after analyzing the vibration of a beam with distributed mass:

1. $\frac{f_1}{f_2} \approx 0{,}1596$
2. Equation (30)

$$E = {f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right) \left(\frac{48\pi^2}{{\delta_n}^4}\right)$$

We will test these two results with simulation from COMSOL Multiphysics software.

3.4.1.1 Eigen frequency ratios

In order to verify these two ratios in COMSOL, we model a beam that is made of Aluminum, fixed at one end and free to move at the other end. This is the exact condition that we set for the theory in section 3.2.

The dimension of the beam is changed in each experiment to prove that the result is reliable. The experiment can be done unlimited time, however, in this task, five experiments are performed.

The process and results are given below:

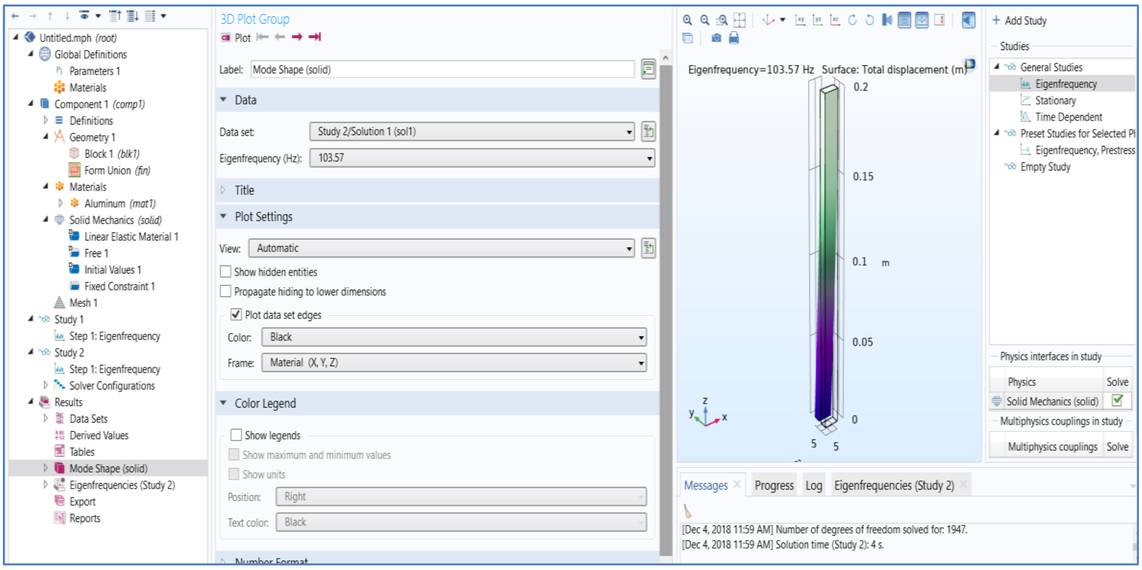

Figure 11: Eigen frequency simulation

Material Contents

	Property	Variable	Value	Unit	Property group
✓	Density	rho	2700[kg/...	kg/m³	Basic
✓	Young's modulus	E	70e9[Pa]	Pa	Young's modulus and Poisso...
✓	Poisson's ratio	nu	0.33	1	Young's modulus and Poisso...
	Relative permeability	mur_is...	1	1	Basic
	Heat capacity at constant pressure	Cp	900[J/(kg...	J/(kg·K)	Basic
	Thermal conductivity	k_iso ; k...	238[W/(...	W/(m·K)	Basic
	Electrical conductivity	sigma_i...	3.774e7[S...	S/m	Basic
	Relative permittivity	epsilon...	1	1	Basic
	Coefficient of thermal expansion	alpha_i...	23e-6[1/K]	1/K	Basic
	Murnaghan third-order elastic m...	l	-2.5e11[Pa]	N/m²	Murnaghan
	Murnaghan third-order elastic m...	m	-3.3e11[Pa]	N/m²	Murnaghan
	Murnaghan third-order elastic m...	n	-3.5e11[Pa]	N/m²	Murnaghan

Figure 12: Mechanical properties of Aluminum attained from COMSOL

The table below shows the results of 5 experiments:

Table 8: Eigen frequency's ratio and error

Width w(m)	**Thickness t(m)**	**Length L(m)**	**First mode frequency f_1(Hz)**	**Second mode frequency f_2(Hz)**	**Ratio f_1/f_2**	**Error (%)**
0,01	0,005	0,2	103,57	647,95	0,159	0,151
0,01	0,0075	0,2	156,29	975,77	0,160	0,357
0,01	0,01	0,2	208,83	1298,5	0,160	0,766
0,015	0,01	0,2	209,54	1305,9	0,160	0,536
0,02	0,01	0,3	92,218	576,31	0,160	0,259

The error (%) shown in table 8 is calculated by:

$$error = \left(\left|\frac{FEA\ value}{theoretical\ value} - 1\right|\right) \times 100\%$$

The errors calculated are relatively small which proves that COMSOL simulation results and theoretical value are close as we desire.

3.4.1.2 Equation of Dynamic Bending modulus formula of a beam with distributed mass

The equation that is mentioned here is equation (30):

$$E = {f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right) \left(\frac{48\pi^2}{{\delta_n}^4}\right)$$

We simplify the equation by examining only the case with the first mode of vibration, which implies that the value of δ_n is 1.875 equation (30) becomes:

$$E = 38{,}33{f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right)$$

Remind that this equation is based on the boundary condition such that the beam is fixed at one end and free to move at the other end. This result is different from equation (1) in section 2.1.2.

$$E = 0{,}9465\left(\frac{mf_f^2}{w}\right)\left(\frac{L^3}{t^3}\right)R_1$$

The reason for this difference is the boundary condition. In equation (1), the sample is clamped at both ends instead of being clamped at only one end.

Some values of R_1 are given in table 1. It is clear that $R_1 \approx 1$. Then the difference between the frequency f_n achieved in equation (30) and the frequency f_f in equation (1) is:

$$\frac{f_n}{f_f} = \sqrt{\frac{0{,}9465}{38{,}33}} = 0{,}157$$

This means that f_n is 6,36 times smaller than f_f. This implies that the beam vibrates 6,36 times slower in one-end fixed experiment than it does with two-end fixed experiment. And because the damping force is proportional to the velocity of the motion, the one-end fixed experiment will undergo smaller damping force which leads to less error in the calibration result.

Now go back to equation (30), we want to add some more transformation. Notice that $m = V\rho = wtL\rho$ where V, ρ are the volume and density of the beam respectively. Replace this to the above equation and solve for f_n we get:

$$E = 38{,}33{f_n}^2\left(\frac{L}{t}\right)^3 tL\rho = 38{,}33{f_n}^2\rho\frac{L^4}{t^2}$$

$$f_n = \frac{0{,}162\sqrt{\frac{E}{\rho}}\,t}{L^2} \tag{43}$$

By setting up similar model and simulation which have been done in section 3.4.1.1 and varying the factors L, t and ρ, E (by adjusting the material), it is possible to observe the relationship between f_n and these elements. And thus, we would verify whether COMSOL gives the result that is the same as theoretical result.

Notice that E and ρ are dependent on the material, thus they are dependent on each other. For that reason, there is no choice but examining the behavior of the whole factor $\sqrt{\frac{E}{\rho}}$ and f_n.

In this set of experiments, it is enough to observe only the first mode (f_1) of vibration. The results are listed below:

1. Dependence of frequency on beam’s length

The width and thickness of the beam are 0,02m and 0,01m respectively.

Table 9: Dependence of frequency on beam's length

Material	**Length L(m)**	**$1/L^2$ (m^{-2})**	**Frequency f_n(Hz)**
Aluminum	0,2	25	208,77
Aluminum	0,25	16	133,37
Aluminum	0,3	11,111	92,218
Aluminum	0,35	8,163	67,688
Aluminum	0,4	6,25	51,785

The graph below shows the relationship between the value of f_n and $\frac{1}{L^2}$.

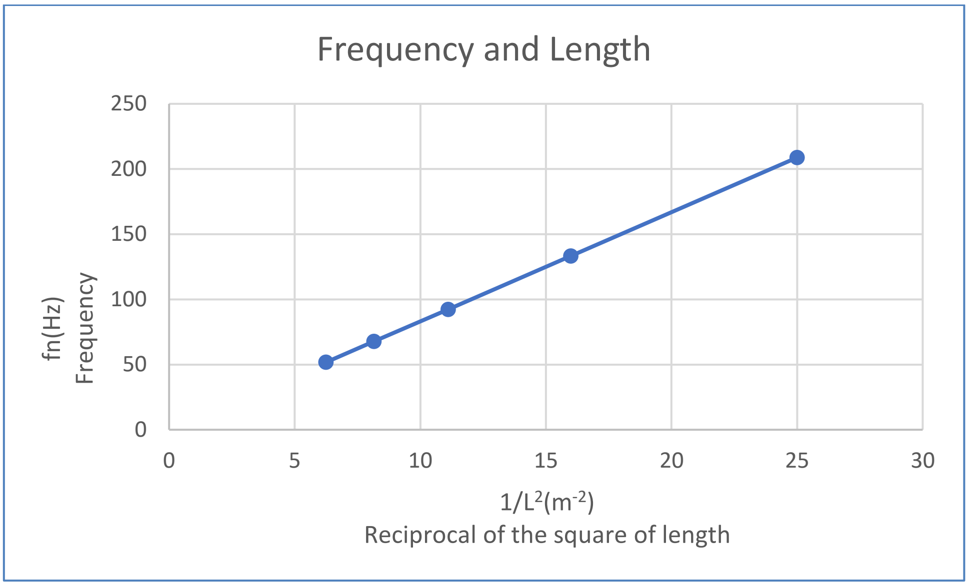

Figure 13: Dependence of frequency on beam's length

The linear graph pointing toward the origin proves that f_n and $\frac{1}{L^2}$ has a linear relationship. The error will be bigger if the damping is significant, that is for short beams.

2. Dependence of frequency on beam's thickness

Table 10: Dependence of frequency on beam's thickness

Material	**Mass w(m)**	**Thickness t(m)**	**Length L(m)**	**Frequency f_n(Hz)**
Aluminum	0,01	0,05	0,2	103,57
Aluminum	0,01	0,06	0,2	124,85
Aluminum	0,01	0,07	0,2	145,74
Aluminum	0,01	0,08	0,2	166,81
Aluminum	0,01	0,09	0,2	188,81

The graph below shows the relationship between f_n and t.

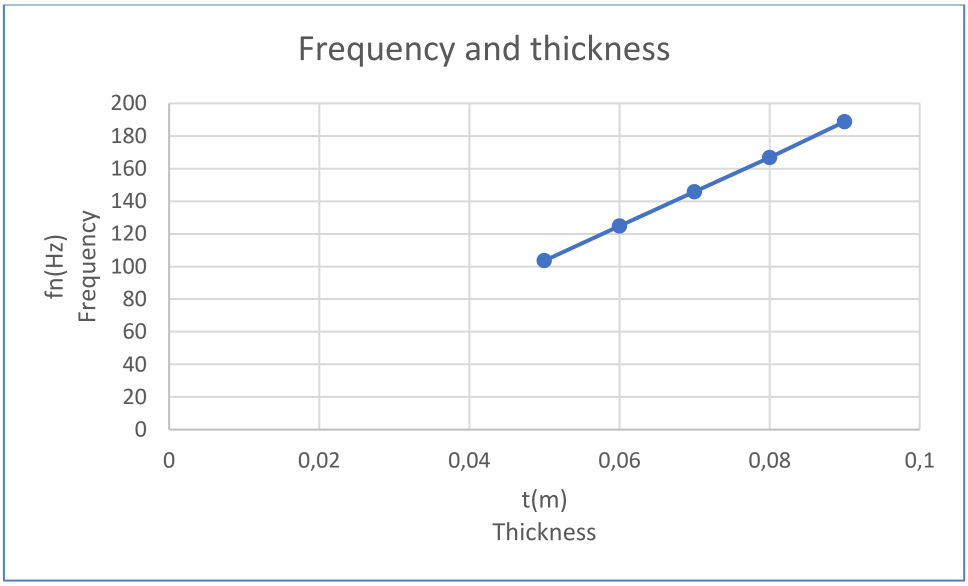

Figure 14: Dependence of frequency on beam's thickness

The linear line tending to pass through the origin proves that f_n and t are linearly dependent as desire.

3. Dependence of frequency on beam's material (variation on E and ρ)

The length, width and thickness of the beam are 0,2m, 0,02m and 0,01m respectively.

Table 11: Dependency of frequency on beam's material

Material	**Young's modulus E(Pa)**	**Density ρ (kg m⁻³)**	**Frequency f_n(Hz)**	$\sqrt{\frac{E}{\rho}}$ **($Pa^{1/2}$ $kg^{-1/2}$ $m^{3/2}$)**
Aluminum	7,00E+10	2700	208,77	5091,751
Cast iron	1,40E+11	7000	181,93	4472,136
Copper	1,10E+11	8960	144,06	3503,824
Iron	2,00E+11	7870	205,78	5041,127
Steel	2,05E+11	7850	208,41	5110,249

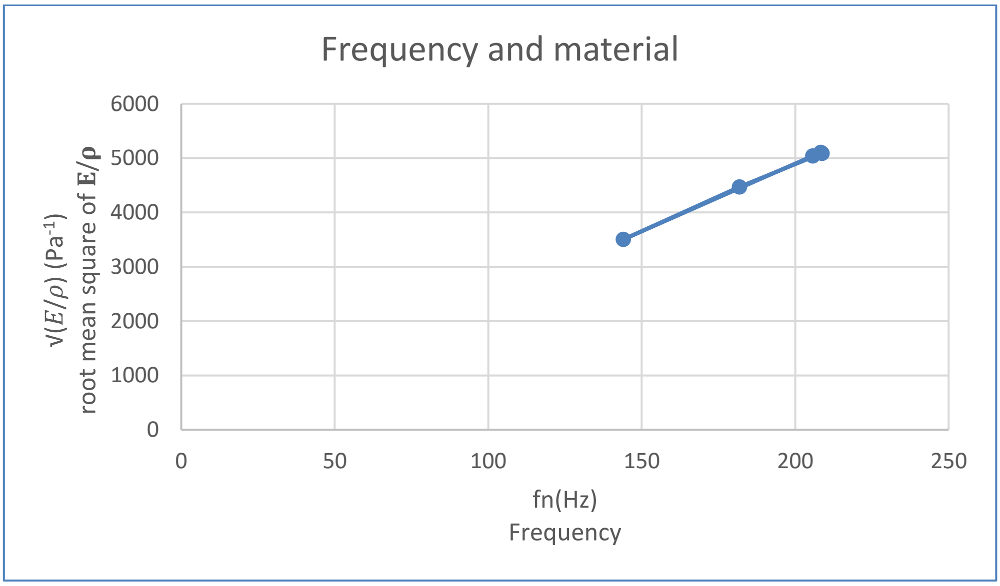

Figure 15: Dependence of frequency on beam's material

The linear line pointing at the origin proves that f_n and $\sqrt{\frac{E}{\rho}}$ are directly proportional as we desire.

We have now proved that f_n is proportional to L, t and $\sqrt{\frac{E}{\rho}}$, which implies that:

$$f_n = \Delta_1 \frac{\sqrt{\frac{E}{\rho}}\,t}{L^2}$$

Where Δ_1 is unknown constant.

However, by using Table 7, the value of Δ_1 can be found. The equation is used beam theory. The thickness must be comparable to width.

Remind that the length, width and thickness of the beam are 0,2m, 0,02m and 0,01m respectively.

Table 12: Constant value

Material	**Young's modulus E(Pa)**	**Density ρ(kg m^{-3})**	**Frequency f_n(Hz)**	$\sqrt{\frac{E}{\rho}}$	**Constant Δ_1**	**error (%)**
Aluminum	7,00E+10	2700	208,77	5091,751	0,164	1,239
Cast iron	1,40E+11	7000	181,93	4472,136	0,163	0,446
Copper	1,10E+11	8960	144,06	3503,824	0,164	1,519
Iron	2,00E+11	7870	205,78	5041,127	0,163	0,791
Steel	2,05E+11	7850	208,41	5110,249	0,163	0,698

The error (%) is calculated by:

$$error = \left(\left|\frac{COMSOL\ result}{0{,}162} - 1\right|\right) \times 100$$

The errors shown in the table reveals that the difference between the COMSOL results and the theoretical solution of the constant is relatively small and negligible.

3.4.2 Eigen frequency for torsional motion

3.4.2.1 Verification of frequency equation of torsional motion for a beam with distributed mass and fixed at one end

In the former section, we have seen the reliable capability of COMSOL software to check and verify the theoretical equation that is built in section 3.2, equation (30).

$$E = {f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right) \left(\frac{48\pi^2}{{\delta_n}^4}\right)$$

This equation is solved for f_n in equation (43):

$$f_n = \frac{0{,}162\sqrt{\frac{E}{\rho}}\,t}{L^2}$$

In this section, by using the same software COMSOL, we will build a formula for the frequency of the beam with distributed mass in the case of torsional motion and with the boundary condition such that the beam is fixed at one end and free at the other end.

In the first place, it is worth noticing that in section 3.3, the formula for frequency of torsional harmonic oscillator has been built in equation (41):

$$\omega_N = \sqrt{\frac{12 \times JG}{Lml^2}} = \sqrt{\frac{4wt^3G}{Lml^2}}$$

This case is different from the case that we are finding because of the boundary condition. Equation (41) is true for the beam fixed at one end and the other end is attached to a cylinder bar that has relatively large mass compared to the mass the beam.
However, because of the primary nature of the two motions is torsion, we expect that the equation that we are trying to develop is somewhat similar to equation (41). In more detail, it is expected that the unknown equation has a form as below.

$$f_t = \sqrt{\frac{4wt^3G}{LY}}$$

Where:

Table 13: Terms and definition

Term	Definition	Unit
f_t	Torsional frequency	Hz
w	With of the beam	m
t	Thickness of the beam	m
G	Shear Modulus	Pa
L	Length	m
Υ	Unknown factor	To be determined

The unknown factor Υ is predictably same as the moment of inertia, which implies $\Upsilon \propto ml^2$.

In simulation to testify the dependence of frequency on length, width and thickness, Aluminum is used.

1. Verification of the dependence on thickness

In this group of simulation, all other factors of the beam including length, width, material (which means density and Shear modulus) are fixed. It is only the thickness of the beam is varied. In this way, the dependence between the torsional frequency and the thickness of the beam can be verified.

The table and graph below show the result:

Table 14: Dependence of frequency on beam's thickness

Width w(m)	Thickness t(m)	Length L(m)	Frequency f t (Hz)
0,05	0,005	0,5	312,46
0,05	0,007	0,5	430,51
0,05	0,009	0,5	543,88
0,05	0,011	0,5	653,52
0,05	0,013	0,5	754,89

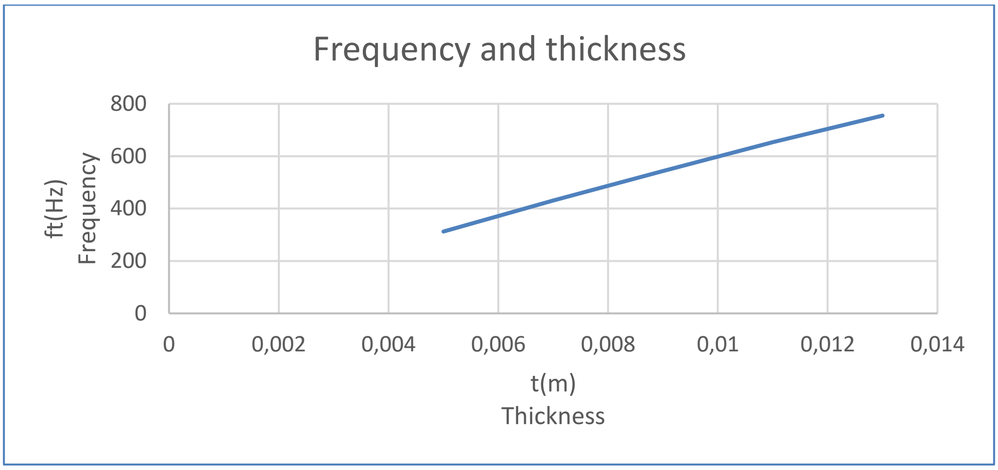

Figure 16: Dependence of frequency on beam's thickness

The straight line pointing at the origin proves that the value frequency is directly proportional to the value of the beam's thickness when other factors are fixed.

2. Verification of the dependence on length

With the same method, the length of the beam is varied and other factors are fixed, it is possible to observe the relationship between torsional frequency of the beam and its length.

Table 15: dependence of frequency on beam's length

Width **w(m)**	**Thickness** **t(m)**	**Length** **L(m)**	**Frequency** **f_t (Hz)**	**Reciprocal of length** **1/L (m^{-1})**
0,05	0,005	0,2	814,63	5
0,05	0,005	0,3	530,02	3,333
0,05	0,005	0,4	393,15	2,5
0,05	0,005	0,5	312,46	2
0,05	0,005	0,6	259,32	1,667

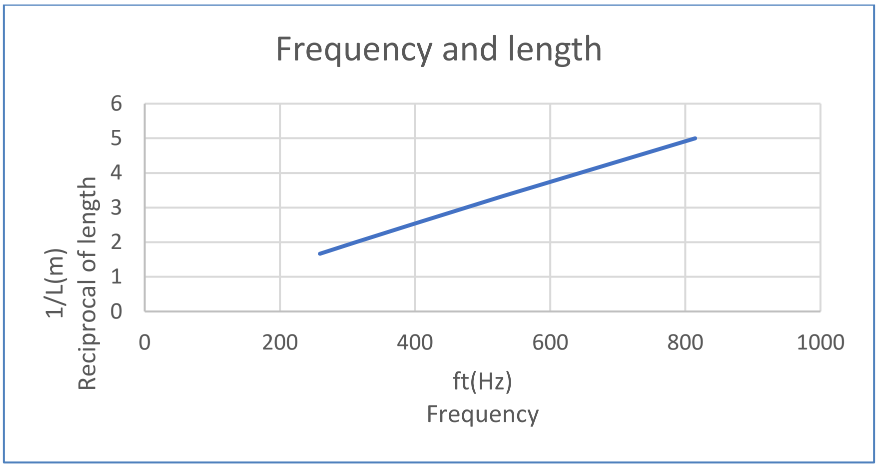

Figure 17: Dependence of frequency on beam's length

The straight line pointing at the origin proves that f_t and $\frac{1}{L}$ are directly proportional.

3. Verification of the dependence on width

Table 16: Dependence of frequency on beam's width

Width **w(m)**	**Thickness** **t(m)**	**Length** **L(m)**	**Frequency** **f_t (Hz)**	**Reciprocal of width** **1/w (m^{-1})**
0,02	0,02	0,5	309,99	50
0,04	0,005	0,5	158,43	25
0,06	0,005	0,5	107,06	16,667
0,08	0,005	0,5	81,257	12,5
0,1	0,005	0,5	65,786	10

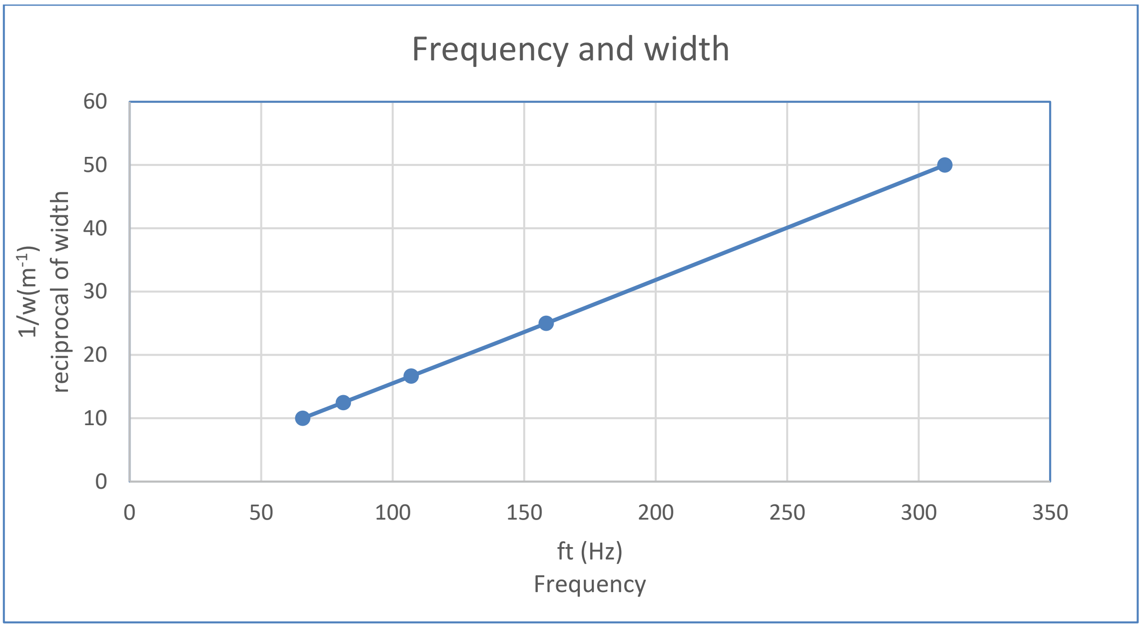

Figure 18: Dependence of frequency on beam's thickness

As can be seen on the graph, the value of the torsional frequency and reciprocal of the width are directly proportional.

4. Verification of the dependence on $\frac{G}{\rho}$

The ratio $\frac{G}{\rho}$ is the ratio between the shear modulus of the material and its density. This ratio is the same the ratio $\frac{E}{\rho}$ that appears in section 3.4.1.2 in equation (43).

In order to see the dependence between f_t and $\frac{G}{\rho}$, simply fixed the dimension of the beam and change only its material.

The table and graph below show the relationship:

The length, width and thickness of the beam are 0,5m, 0,05m and 0,005m respectively.

Table 17: Dependence of frequency on beam's material

Material	**Density** ρ **(kg m^{-3})**	**Shear modulus G (Pa)**	**Frequency f_t (Hz)**	G/ρ **(Pa kg^{-1} m^3)**	**f_t^2 (s^{-2})**
Aluminum	2700	26315789474	312,46	9746589	97631,252
Cast iron	7000	56000000000	282,56	8000000	79840,154
Copper	8960	40740740741	213,54	4546958	45599,332
Iron	7870	77519379845	313,8	9849985	98470,44
Steel	7850	80078125000	319,27	10201035	101933,333

Table 18: Dependence of frequency on beam’s material

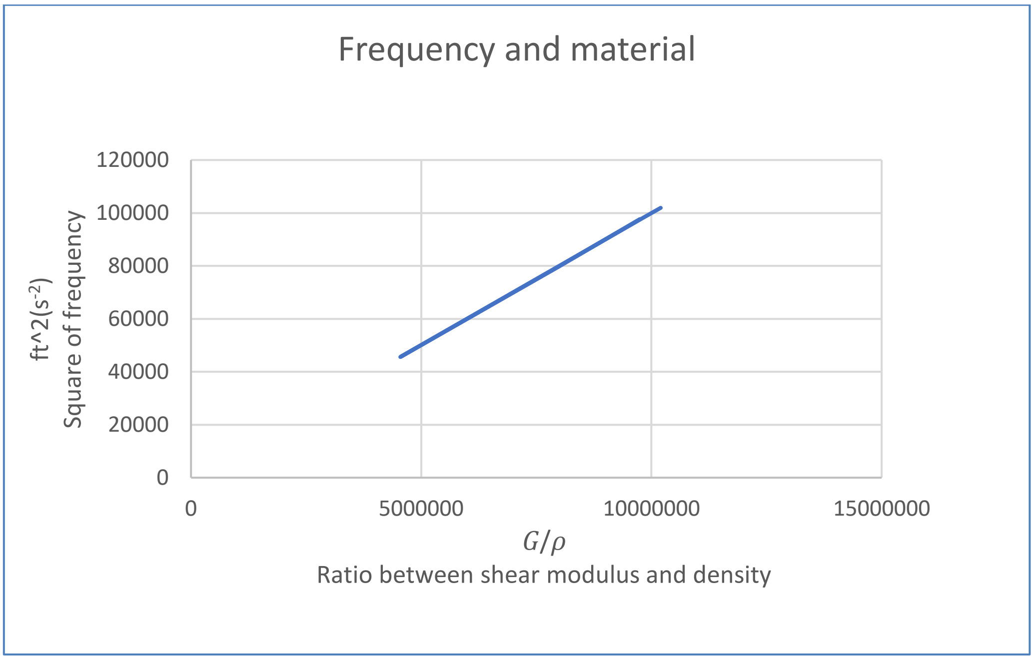

Figure 19: Dependence of frequency on beam’s material

The graph proves that ${f_t}^2$ is proportional to $\frac{G}{\rho}$ implying f_t is proportional to $\sqrt{\frac{G}{\rho}}$.

5. Formula of the frequency

We have proved that f_t is proportional to $t, \frac{1}{L}, \frac{1}{w}, and\ \sqrt{\frac{G}{\rho}}$. From this we have:

$$f_t = \Delta_t t \frac{1}{L} \frac{1}{w} \sqrt{\frac{G}{\rho}}$$

Where Δ_t is a constant that can be found by simply solve the above equation for Δ_t and use that simulation that have been done previously.

The length, width and thickness of the beam are 0,5m, 0,05m and 0,005m respectively.

Table 19: Constant value

Material	**Frequency** f_t **(Hz)**	G/ρ **(Pa kg^{-1} m^3)**	**Constant** Δ_t
Aluminum	312,46	9746589	0,5004
Cast iron	282,56	8000000	0,4995
Copper	213,54	4546958	0,5007
Iron	313,8	9849985	0,4999
Steel	319,27	10201035	0,4998

As can be seen on the table $\Delta_t \approx$ 10. We get:

$$f_t = 0{,}5\frac{t}{wL}\sqrt{\frac{G}{\rho}} \tag{44}$$

Now we proceed with some transformation from equation (45):

$$f_t = 0{,}5\sqrt{\frac{G}{\rho}\left(\frac{t}{wL}\right)^2} = 0{,}5\sqrt{\frac{wt^3G}{(wLt\rho)Lw^2}} = 0{,}5\sqrt{\frac{wt^3G}{Lmw^2}}$$

$$f_t = 0{,}5\sqrt{\frac{wt^3G}{12L(\frac{1}{12}mw^2)}} \tag{45}$$

If $w \gg t$ then we have $w^2 \approx w^2 + t^2$ then equation (46) becomes:

$$f_t = 0{,}5\sqrt{\frac{\frac{1}{3}wt^3G}{4L(\frac{1}{12}m(w^2+t^2)}} = 0{,}25\sqrt{\frac{JG}{LI_L}}$$

$$f_t = 0{,}25\sqrt{\frac{JG}{LI_L}} \tag{46}$$

Where $J = \frac{1}{3}wt^3$ is the number that has been mentioned in section 2.3; I_L is the mass moment of inertia of the beam about the axis that passes through its center of mass and parallel to the its length.

3.4.2.2 Verification of frequency equation of torsional motion for a beam with distributed mass and fixed at two ends

In this sector, we study the torsional vibration of a beam with distributed mass that is fixed at both ends. This case is similar to the former case with the only difference being that both faces are locked instead of one.

We examine this case with the same method and the same factors: thickness, length, width and the ratio $\frac{G}{\rho}$.

1. Dependence of the frequency on thickness

Table 20: Dependence of the frequency on thickness

Material	Width w(m)	Thickness t(m)	Length L(m)	Frequency f_t(Hz)
Aluminum	0,05	0,005	0,5	643,01
Aluminum	0,05	0,006	0,5	766,06
Aluminum	0,05	0,007	0,5	885,76
Aluminum	0,05	0,008	0,5	1005,7
Aluminum	0,05	0,009	0,5	1118,2

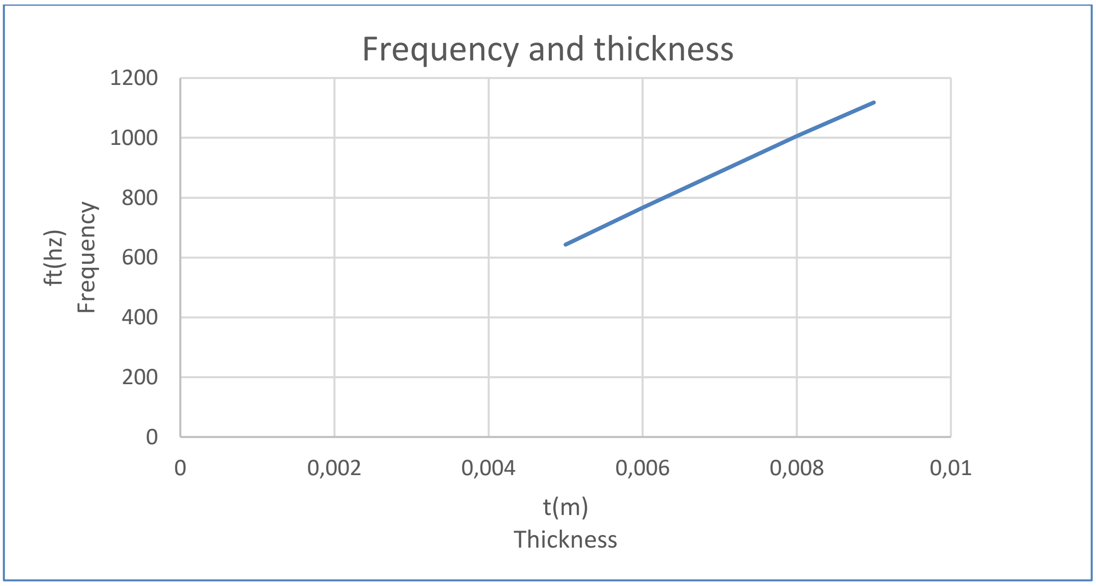

Figure 20: Dependence of the frequency on thickness

The graph shows that the frequency of vibration if directly proportional to the thickness of the beam.

2. Dependence of the frequency on length

Table 21: Dependence of the frequency on beam's length

Material	**Width** **w(m)**	**Thickness** **t(m)**	**Length** **L(m)**	**Frequency** **f_t(Hz)**	**Reciprocal of Length** **1/L(m^{-1})**
Aluminum	0,05	0,005	0,5	643,01	2
Aluminum	0,05	0,005	0,6	530,9	1,67
Aluminum	0,05	0,005	0,7	452,79	1,43
Aluminum	0,05	0,005	0,8	395,1	1,25
Aluminum	0,05	0,005	0,9	349,91	1,11

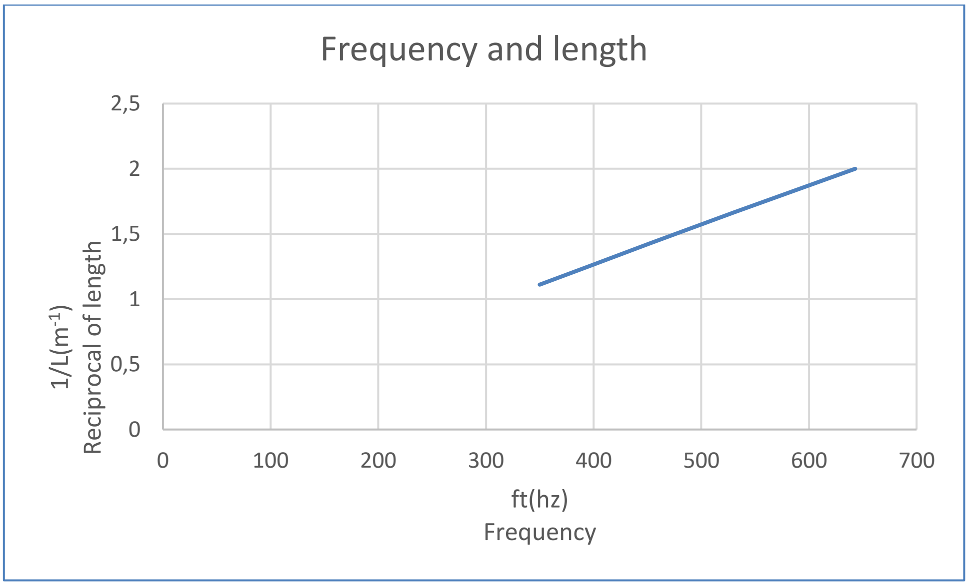

Figure 21: Dependence of the frequency on length

The straight line passing through the origin proves that the value of the frequency is inversely proportional to the length of the beam.

3. Dependence on the width

Table 22: Dependence of frequency on beam's width

Material	**Width** **w(m)**	**Thickness** **t(m)**	**Length** **L(m)**	**Frequency** **f_t(Hz)**	**Reciprocal of width** **1/w(m^{-1})**
Aluminum	0,05	0,005	0,5	643,01	20,00
Aluminum	0,06	0,005	0,5	544,8	16,67
Aluminum	0,07	0,005	0,5	474,58	14,29
Aluminum	0,08	0,005	0,5	421,37	12,50
Aluminum	0,09	0,005	0,5	380,19	11,11

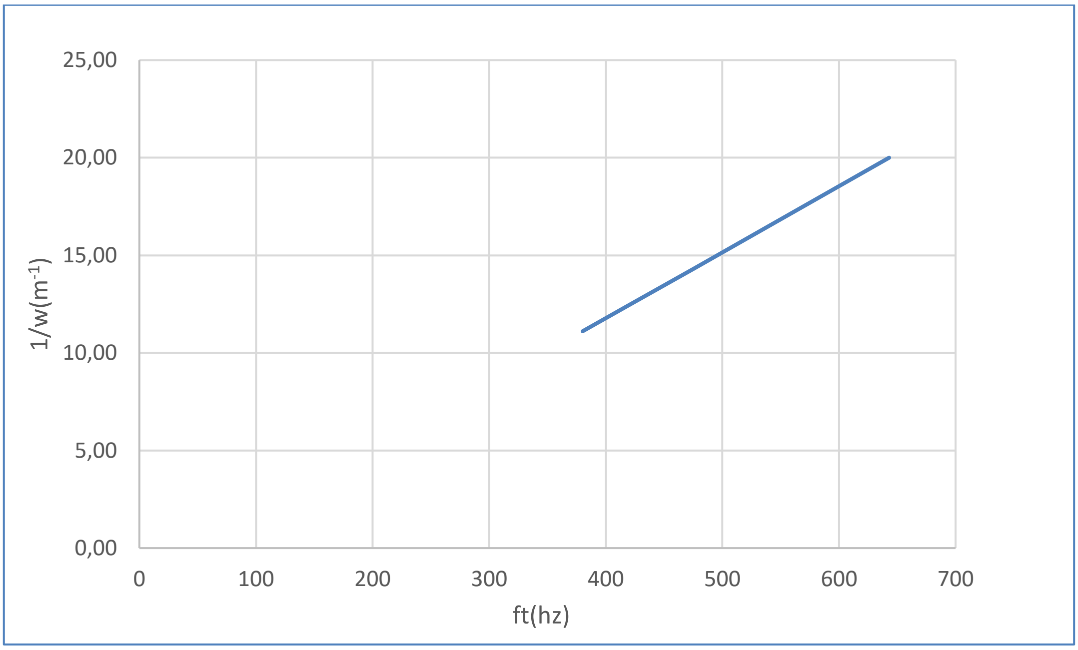

Figure 22: Dependence of the frequency on beam's width

The graph shows that the value of the frequency is inversely proportional to the value of the width.

4. Dependence of the frequency on ratio $\frac{G}{\rho}$

It is the same as in section 3.4.2.1, we examine the dependence of the frequency on the material of the beam. The simulation is done by fixing the dimensions of the beam and have its material varied.

The length, width and thickness of the beam are 0,5m, 0,05m and 0,005m respectively.

Table 23: Dependence of frequency on beams material

Material	$\frac{G}{\rho}$ **(kg m^{-3} Pa^{-1})**	**Frequency f_t (Hz)**	**f_t^2 (Hz^2)**
Aluminum	9,75E+06	643,01	413461,9
cast iron	8,00E+06	580,33	336782,9
Copper	4,55E+06	439,69	193327,3
Silicon	2,85E+07	1097	1203409
Acrylic plastic	9,96E+05	206	42345,41

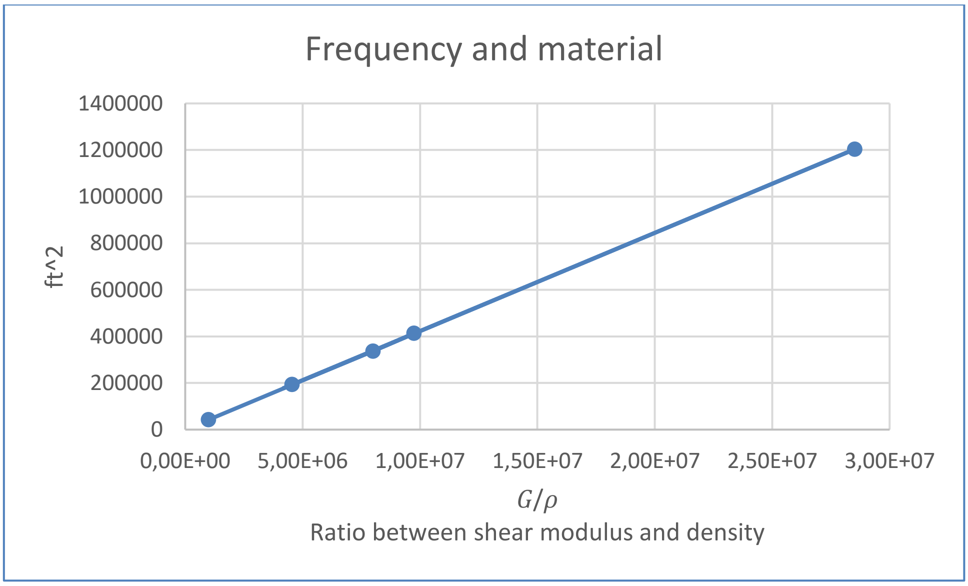

Figure 23: Dependence of frequency on beam's material

The graph shows that the square of the value of the frequency is directly proportional to $\frac{G}{\rho}$, or $f_t \propto \frac{G}{\rho}$.

5. Finding the formula of frequency according to the value of length, thickness, width and ratio $\frac{G}{\rho}$ of the beam.

According to the relationship between the frequency with the length, thickness, width and material of the beam, we have:

$$f_t = \Delta \frac{t}{wL}\sqrt{\frac{G}{\rho}}$$

Solving above equation for Δ gives:

Table 24: Constant value

Material	**Width w(m)**	**Thickness t(m)**	**Length L(m)**	**Frequency f_t(Hz)**	**Constant Δ**
Aluminum	0,05	0,005	0,5	643,01	1,02982
cast iron	0,05	0,005	0,5	580,33	1,025888
Copper	0,05	0,005	0,5	439,69	1,030994
Silicon	0,05	0,005	0,5	1097	1,027204
Acrylic plastic	0,05	0,005	0,5	205,78	1,030988

It is clear that $\Delta \approx 1$, we have:

$$f_t = \frac{t}{wL}\sqrt{\frac{G}{\rho}}$$

And following the same steps as we did to achieve equation (47), we get:

$$f_t = 0{,}5\sqrt{\frac{JG}{LI_L}} \qquad (47)$$

And remind that equation (47) is only true for a thin beam $(t \ll w)$ of which both ends are fixed at both two ends.

3.4.2.3 Verification of frequency equation of torsional vibration of a beam with small distributed mass attached with a large mass in the middle, fixed at both ends

In this section, we will simulate the vibration of a thin beam locked at both ends and attached with a cylinder object that has the mass relatively large compare to the mass of the beam.

Then by using the same method as in other sections, we attempt to find the formula of the frequency of vibration in term of other known factors of the beam and the object.

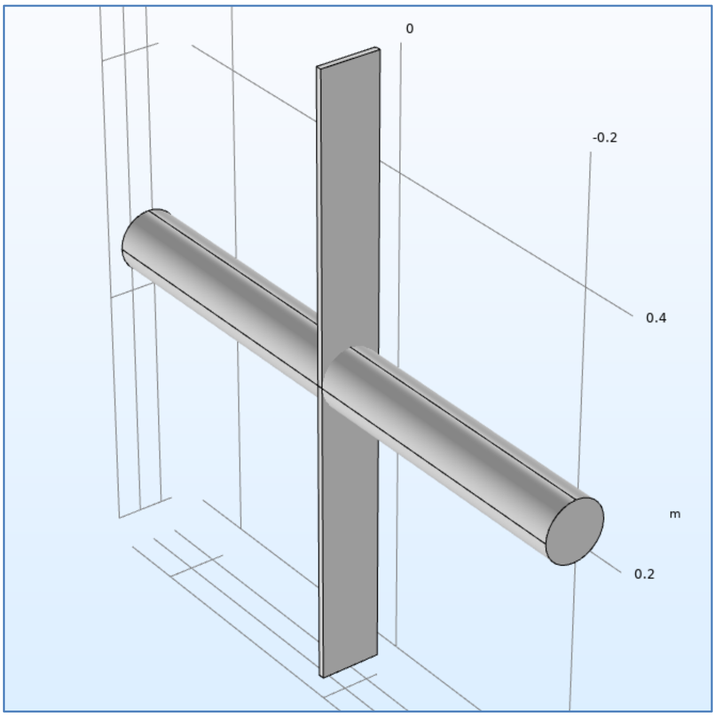

Figure 24: Cylinder object attached to thin beam

The factors that are concerned in this simulation are the dimensions and material of the beam of attached object.

1. Dependence of the frequency on the length of the beam

In the table below, l is the length of the cylinder object and R is the radius.

Table 25: Dependence of frequency on beam's length

l(m)	**R(m)**	**w(m)**	**t(m)**	**L(m)**	**f_t(Hz)**	$\frac{1}{L}$ **(m^{-1})**	$\frac{1}{\sqrt{L}}(m^{-\frac{1}{2}})$	$\frac{1}{L^{\frac{3}{2}}}(m^{-\frac{3}{2}})$
0,5	0,025	0,05	0,005	0,5	1497,5	2	1,414214	1,587
0,5	0,025	0,05	0,005	0,6	1201,4	1,667	1,291	1,406
0,5	0,025	0,05	0,005	0,7	1003,1	1,429	1,195	1,268
0,5	0,025	0,05	0,005	0,8	863,33	1,250	1,118	1,160
0,5	0,025	0,05	0,005	0,9	757,49	1,111	1,054	1,072

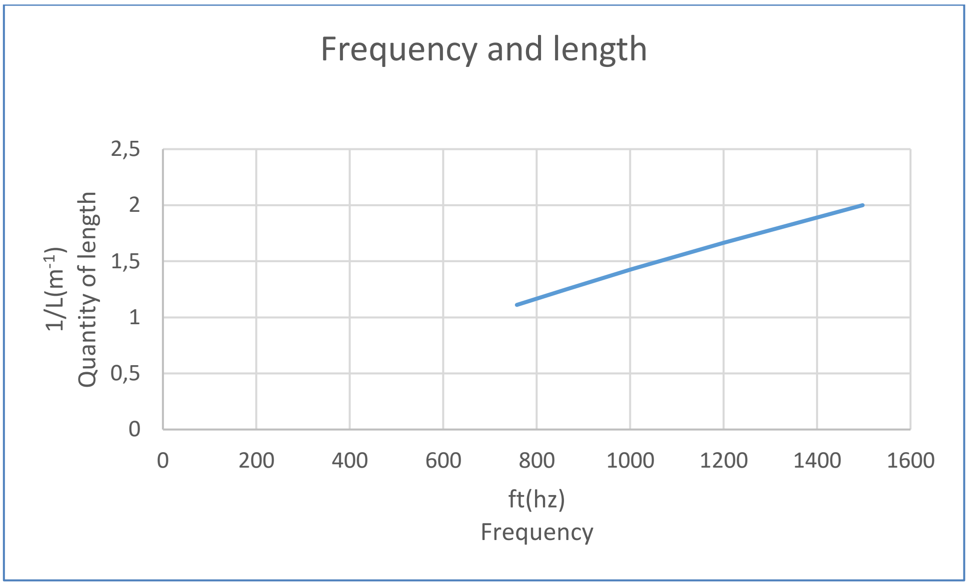

Figure 25: Relationship between frequency and beam's length

It is seen from the graph that the frequency is proportional to the reciprocal of the beam's length, however there is no linear relationship between them.

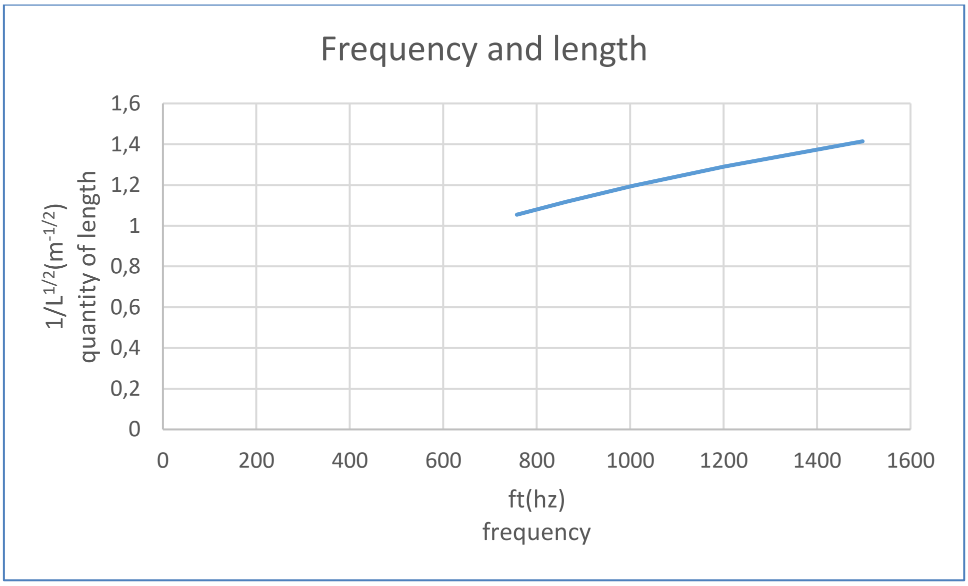

Figure 26: Relationship between frequency and beam's length

In the above graph, we replace $\frac{1}{L}$ by $\frac{1}{\sqrt{L}}$ to examine the relationship with the frequency. It is clear that $\frac{1}{\sqrt{L}}$ and f_t are proportional but do not have linear relationship.

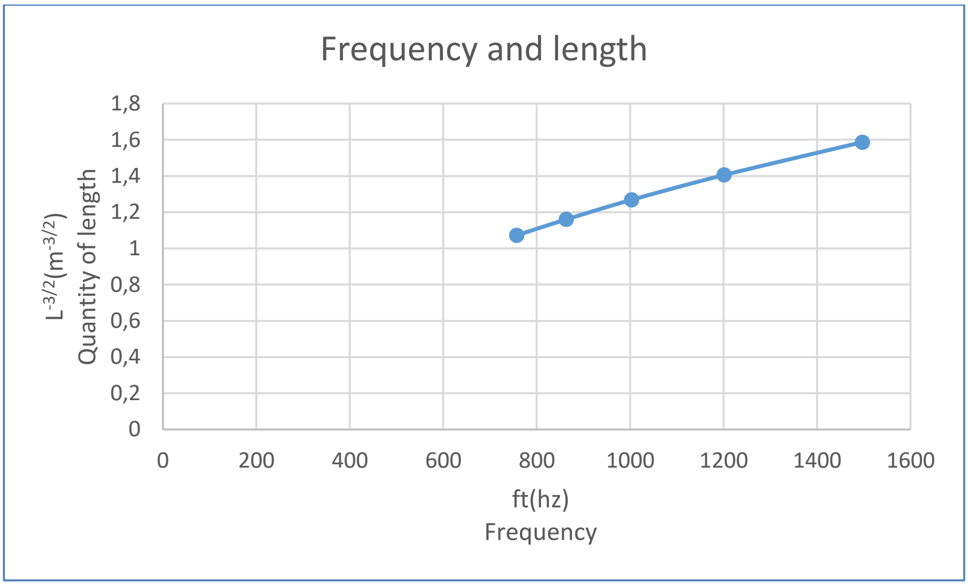

Figure 27: Relationship between frequency and beam's length

We examine another variation as above. Again, $\left(\frac{1}{L}\right)^{\frac{3}{2}}$ and f_t are proportional but not directly proportional.

As can be seen in this case, the frequency of the beam is clearly dependent on the length of the beam. However, it fails to find a clear relationship between them. This leads to the fact that it is not feasible this time to find a formula as we desire.

Thus, in this case, we conclude that it is not possible to use COMSOL multiphysics to achieve a formula for the frequency.

4 EXPERIMENTS

In section 3.6, COMSOL Simulation is used to help developing formulas and testify the theory in the cases of torsional vibration of cantilever beam with distributed mass and one end fixed, and torsional motion of cantilever beam with distributed mass and two ends fixed. As discussed at the end of section 3.4.2.3, the software is incapable of giving a clear formula for the frequency of torsional vibration in the case where two ends of the beam are fixed.

The experiments are implemented in an attempt to, again, testify the formula of the frequency of a cantilever beam with distributed mass fixed at one end undergoing translational vibration, which is the equation (30), $E = {f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right) \left(\frac{48\pi^2}{1{,}875^4}\right)$ which is the other form of equation (29), $f_n = \frac{\omega_n}{2\pi} = \frac{{\delta_n}^2}{2\pi L^2}\sqrt{\frac{EI_A}{\rho A}}$ with $\delta_n = 1{,}875$.

In addition to this, one apparatus is built in order to verify the frequency of vibration of a cantilever beam undergoing torsional vibration that is attached to a relatively big mass.

The first experiment is successfully conducted and showed a similar result as described in the theory and to the bending modulus achieved from static 3 point bending test that has been done simultaneously. However, torsional vibration test has not been conducted because of the lack of equipment at Arcada.

4.1 Apparatus for bending modulus testing

4.1.1 Procedure

The specimen is made of carbon fiber lamina with the free length being 149mm, the width being 15mm and thickness being 2,25mm.

The original length of the specimen is 180mm. The free length is the length of the part of the specimen that undergoes vibration.

The specimen is clamped at one end and allowed to vibrate at the other end. A force is initiated to cause the vibration, after which the spectrum of the frequency of vibration is achieved.

By reading the frequency's spectrum, the first mode of frequency of translational vibration is calculated.

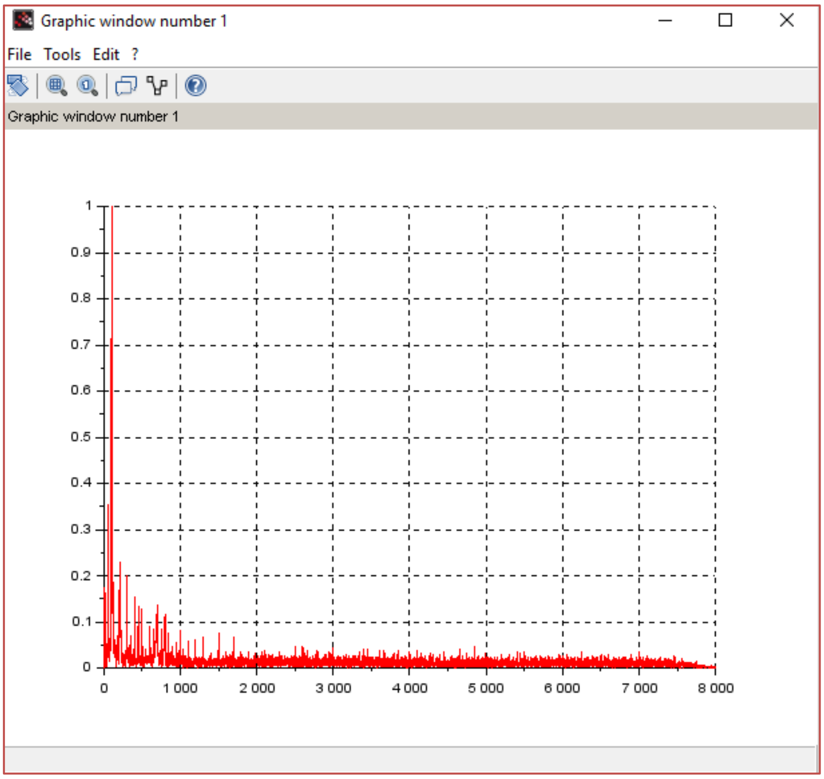

Figure 28: spectrum of frequency

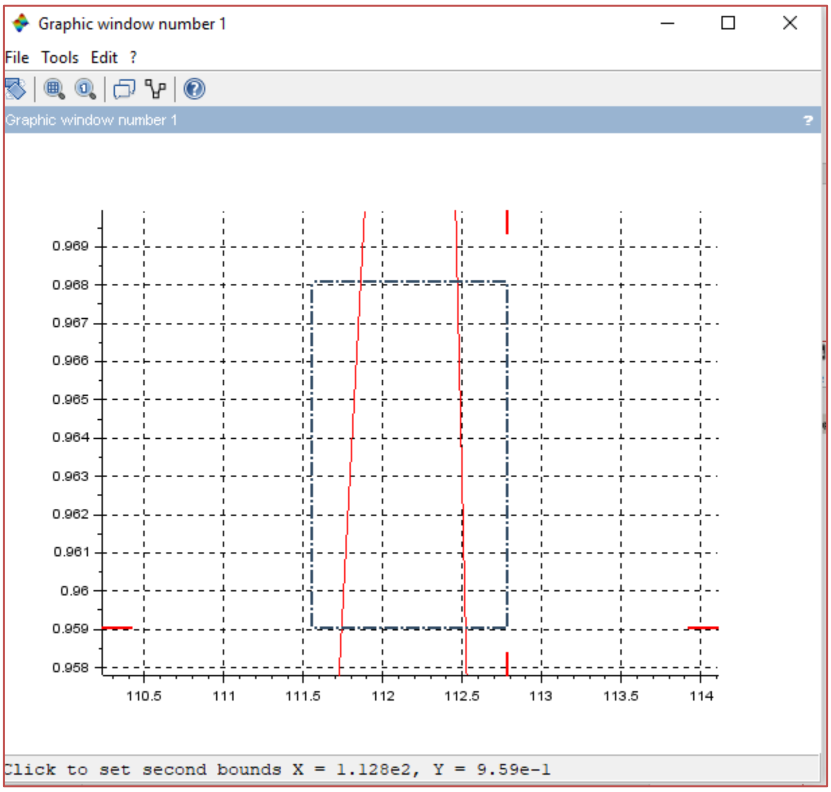

Figure 29: Reading of frequency

After achieving the frequency, static Young modulus, *E*, is calculated by solving equation (29) for *E*.

Finally, the specimens are sent to dynamic 3 point bending test in Arcada laboratory and Borås laboratory in order to calculate the dynamic Young modulus according to this method. Then the results are recorded and compared.

4.1.2 Result

It is known that the thickness of the sample is 2,25m; the free length of the sample is 149mm; it is then calculated that the total mass of the sample is 8,57g and the mass of the free part is 7,094g.

Table 26: Test results [12]

item	I (mm^4)	Young's Modulus calculated by equitation (29) (GPa)	Young's Modulus calculated at Arcada's laboratory (GPa)	Young's Modulus calculated at Borås' laboratory (GPa)
CCC 2.1	14,23	78,52	77,21	75,96
CCC 2.2	14,23	64,23	77,21	75,969
CCC 2.3	14,23	79,77	77,21	75,96

As can be seen from the table, the error of the Young's modulus from the static bending test at Lab Arcada and Lab Borås is considerably small:

$$Error = \left|\left(1 - \frac{77{,}21}{75{,}96}\right) \times 100\%\right| \approx 1{,}6\%$$

In the same way, we can calculate the error of the Young's modulus calculated by equation (29) with each sample and the Young's modulus calculated at Arcada's laboratory. For example, with sample CCC 2.1, we have:

$$Error = \left|\left(1 - \frac{78{,}52}{77{,}21}\right) \times 100\%\right| \approx 1{,}697\%$$

Apply this calculation for both three samples, we get the table showing errors between the results.

Table 27: Error

Specimen	Error compared to Lab Arcada result (%)	Error compared to Lab Borås result (%)
CCC 2.1	1,697	3,3702
CCC 2.2	16,811	15,4423
CCC 2.3	3,316	5,0158

4.2 Apparatus for shear modulus testing

As mention above, the apparatus for shear modulus testing is not possible to produce at Arcada laboratory. However, it is expedient to recommend an effective set-up for this test The model below gives a good figure of the recommended apparatus.

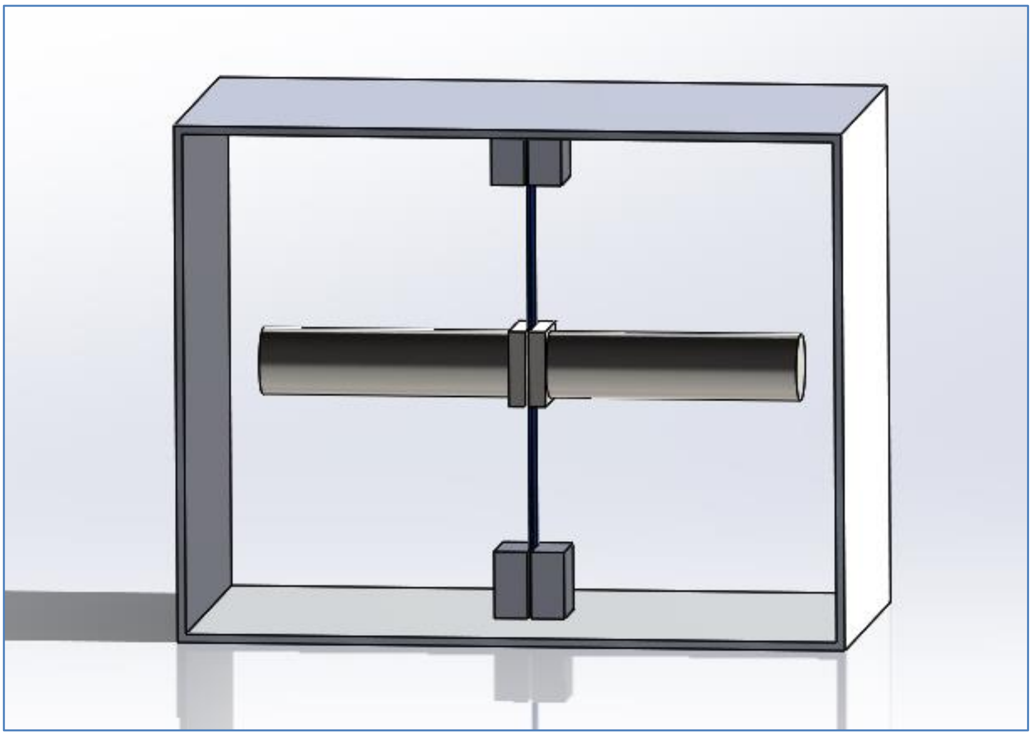

Figure 30: Modelling of test's apparatus

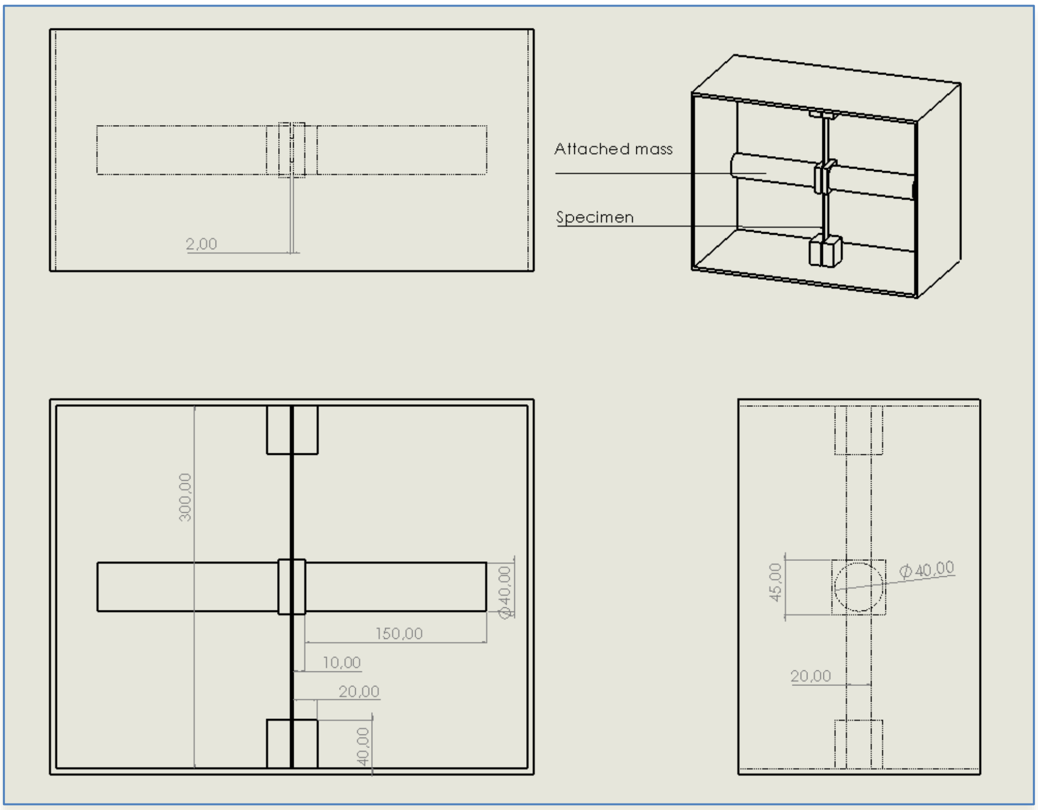

Figure 31: Schematic drawing

1. Prepare a specimen of carbon fiber lamina with the length being 300mm, thickness being 2mm and width being 20mm.
2. Place the specimen inside an aluminum frame. Clamp both ends of the specimen.
3. Clamp two part of cylinder-shape steel mass at the middle of the specimen as Figure 30.
4. Swing the mass in the horizontal direction. Calculate the frequency simply by counting of one revolution as the speed of vibration is very small.

After defining the frequency of vibration, we would expect to use equation (47) to calculate the dynamic shear modulus:

$$f_t = 0{,}5\sqrt{\frac{JG}{LI_L}}$$

However, there should be variation in this case as there is a big mass attached to the specimen. It is predicted that the value of the inertia in this case is the sum of the value of the inertia of both the extra mass and the specimen instead of merely the value of inertia of the specimen as can be seen in equation (47). In addition to this the constant coefficient in this case is expectedly not 0,5.

5 RESULTS

The study of the thesis has been done on six cases in which five cases give a formula for the frequency as a function of chosen factors such as length, width, thickness and material. In each case, the set-up of the apparatus is different which leads to different result.

1. Case 1: Cantilever beam with external mass (section 3.1)

In this case, a beam is fixed at one end and attached with an extra mass at the other end. The weight of the beam is negligible compared to the weigh of the extra mass. The vibration is translational and gives dynamic Young's modulus.

$$E = 16\pi^2 {f_N}^2 \left(\frac{m}{w}\right) \left(\frac{L}{t}\right)^3$$

This case has been done theoretically.

2. Case 2: Free vibration of Cantilever beam (section 3.2)

In this case, a beam is fixed at one end and allowed to vibrate at the other end. The weight of the beam is considered. The vibration is translational and gives dynamic Young's modulus.

$$f_n = \frac{\omega_n}{2\pi} = \frac{{\delta_n}^2}{2\pi L^2}\sqrt{\frac{EI_A}{\rho A}} \quad [9]$$

Especially, in this case, COMSOL simulation has been done as well as practical experiment. Both gives results that testifies the reliability of the equation.

3. Case 3: Torsional harmonic motion (section 3.3)

In this case, a beam is attached with a cylinder-shaped extra mass at one end and is clamped at the other the end. The weigh of the beam is negligible compared to the weigh of the extra mass. The motion is torsional and harmonic from which shear modulus has been found.

$$G = \frac{Lm{f_N}^2}{wt} \times \frac{\pi^2 l^2}{t^2}$$

This case has been done theoretically.

4. Case 4: Torsional vibration of a beam fixed at one end (section 3.4.2)

In this case, a beam is clamped at one end and allows to move at the other end. There is no extra mass. The vibration is torsional from which the function of torsional frequency and shear modulus and other factors has been found.

$$f_t = 0{,}25\sqrt{\frac{JG}{LI_L}}$$

This case is done with COMSOL simulation.

5. Case 5: Torsional vibration of a beam fixed at two ends (section 3.4.3)

In this case, a beam is clamped at both ends and undergoing torsional vibration.

$$f_t = 0{,}5\sqrt{\frac{JG}{LI_L}}$$

This case is done with COMSOL simulation.

6 DISCUSSION

In the recommended apparatus, it is necessary to have both ends of the specimen clamped in order to prevent potential translational motion caused by the large extra mass.

In case the recommended apparatus fails to give a good result, it is still possible to modify the apparatus such that the specimen is clamped at one end and attach to a big extra mass at the other end. This means that we accept the risk of error reading due to potential translational motion. If this apparatus is implemented, then equation (42) in section 3,3 will be used to calculate the shear modulus.

$$G = \frac{\pi^2 {f_N}^2 L m l^2}{w t^3} = \frac{L m {f_N}^2}{w t} \times \frac{\pi^2 l^2}{t^2}$$

7 CONCLUSION

The research has successfully developed the theory for calculating dynamic Young and Shear modulus by vibrational method for light-weighed material. This includes theoretical parts developed based on the existing knowledge about beam theory and other previous study on the same subject with heavy-weighed material. In addition to this, Finite Element Method has been used to verify formulas in some other important cases along with practical experiments that are built to strengthen some of these results.

Although the formula for the case in which a big mass is attached to a 2-end-fixed beam has not been verified and the experiment aimed to calculate torsional frequency has not been done due to some particular reasons, the case is still much likely to be solved by the recommended apparatus and its alternative.

Throughout the research, the most reliable formula that has been developed is equation (29), $f_n = \frac{\omega_n}{2\pi} = \frac{{\delta_n}^2}{2\pi L^2}\sqrt{\frac{EI_A}{\rho A}}, with\ \delta_n = 1{,}875$ or its transformation, equation (30), $E = {f_n}^2 \left(\frac{L}{t}\right)^3 \left(\frac{m}{w}\right)\left(\frac{48\pi^2}{1{,}875^4}\right)$ as these equations have been proven both by theory, Finite Element Method and practical experiment. Remind that these equations is used to define translational frequency for a beam that is fixed at one end and vibrating at the other end.

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9 TABLE OF ABBREVIATION AND UNITS

Table 28: Table of abbreviation and units

Term	unit	Definition
E	Pa (kg m^{-1} s^{-2})	Young's modulus
m	kg	Mass
f_f	s^{-1}	Flexural frequency
L	m	Length
w	m	Width
t	m	thickness
R_1	Dimensionless	Correction factor
R_2	Dimensionless	Correction factor
G	Pa (kg m^{-1} s^{-2})	Shear Modulus
f_t	s^{-1}	Torsional frequency
υ	Dimensionless	Poisson's ratio
$\overrightarrow{F_d}$	Newton (kg m s^{-2})	Damping force
b	Dimensionless	Damping constant
v	m s^{-1}	Velocity
a	m s^{-2}	Acceleration
k	Newton per meter (kg s^{-2})	Spring's stiffness
x_m	m	Amplitude of oscillator
e	≈ 2.71828	Euler number
ω_n	Rad s^{-1}	Natural frequency in radian
f_n	Hz (s^{-1})	Natural frequency in Hz
A	m^2	Area
ρ	kg m^{-3}	Density
τ	Pa (kg m^{-1} s^{-2})	Shear stress
T	kg m^2 s^{-2}	Torque
α	Dimensionless	Constant obtained by theory of elasticity listed in Table 1

β	Dimensionless	Constant obtained by theory of elasticity listed in Table 1
θ	Rad	Angle of twist
J	m^4	Torsional constant
G	Pa (kg m^{-1} s^{-2})	Modulus of rigidity or shear modulus
I	kg m^2	Moment of inertia of the bar
I_A	m^4	Area moment of inertia
C	J s rad^{-1}	Angular damping constant
κ	N m rad^{-1}	Torsion spring constant
T	N m	Drive torque
ω_N	rad s^{-1}	Un-damped resonant frequency in radian
ω_d	rad s^{-1}	Damped resonant frequency in radian
ϱ	s^{-1}	Reciprocal of damping time constant
ϕ	rad	Phase angle of oscillation
Δ_1	Dimensionless	Constant value

10 APPENDIX

Tables and figure