Comparision of Deformation for Solid 3D and Beam Models using Finite Element Analysis

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Abstract:

Computation in Engineering Analysis are possible using a software COMSOL Multiphysics. Multiphysics COMSOL is a simulation tool that understands, predicts and optimize based on a physics by using a numerical simulation. The software utilizes the finite element method as a numerical technique. The research paper helps to clarify the best method to test and analyze the material properties of an object. The COMSOL simulation is based on Euler Bernoulli beam model and 3D solid mechanics.

The results are compared based on the theory and experimented results. The material is selected from COMSOL list of materials. The material is predefined and set to be STEEL ASI 4340.The test method is based on three deformations which are: first, uniaxial elongation analyses using a rectangular cross-section geometries by using displacement result: second, the bending deformation due to the force exerted at a midpoint using a rectangular cross-section beam geometries: thirdly, the torsion deformation of the cylindrical beam in which the angle of twist is measured.

The beam model showed a precise accuracy with the analytical computation. In the 3D model, there is error difference in the values when compare to analytical solution. The error value in 3D solid is higher in axial deformation. The time and physical memory are also considered and compared while studying the models. The time and memory cost are better in beam model than 3D solid model.

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1 INTRODUCTION

In the Engineering field one of the major and core practices is performing analysis and simulation before production. In the current market, the pressure has led to an increase demand on performance. Cost and time are the main challenges facing most industries. The globalization and competition have left most industries vulnerable especially with new technologies constantly emerging. A short span and inefficient resources available are the biggest concerns for industries. The industries are finding ways to utilize the resources and become productive using technologies. This has resulted in a high demand in product development and process to enforce time and cost effectiveness in the market. [1]

Solid mechanics is one of the oldest fields of Science. The need to understand material behavior is growing in a multiple scale. That immerging experiment, mathematical models, and simulation tools made possible to advance quick in most sectors and technology. The success in development into different fields is dependent on the advancement of solid mechanics and its application. Solid mechanics is the controlling factor for immerging technologies now and in the future. [2]

The application of solid mechanics is concerned with the analysis of stress and deformation leading to the failure of solid materials. It is the tool used to understand the occurrence of stress and deformation that contributes and improve living conditions. The structural mechanics is a solution for the field of structure and solid mechanics. The structural mechanics has contributed to the world of engineers, researchers and teachers. The tools help the user learn and get the most out of it with a short period of time. The interface encompasses of multi Multiphysics models and methods. [3] [4] [5]

The solid objects show a deformation characteristic due to the force and constraint applied. During the deformation the spatial coordinates changes while the materials coordinates is kept steady. Solid mechanics uses the finite element method to utilize and equate partial differential equations. The technique provide an approximate solution by changing complex finite element to simple finite element. Nowadays, with the availability of different kinds of software's it is possible to analyze and solve different kinds of problems. [6]

Product developments are highly assisted by new computerized software to perform and ease design and analysis. The software not only help on designing but also analyzing and validation prior to manufacturing. Errors related to engineering analysis are being avoided on pre-processing by using computerized simulations. [6]

Computer aided designs gives modeling a quality and insures the user with a flexibility and simplicity. Computer aided design allows for products to be verified virtually. During the designing, the Technologies gives flexibility and freedom for an engineer or researcher to make changes. [1] [7]

1.1 Aims and Objective of thesis

The research is conducted to investigate the structural mechanics models and compared with analytical exact solution. The models are analyzed based on three deformation state for each models. The deformation analysis are: axial elongation, bending and torsional deformation. The geometry used are rectangular cross-sectional and cylindrical shaped models. The rectangular cross-section geometry structure is used for axial elongation and bending deformation analysis. The cylindrical geometry is used for the torsional deformation analyses. The main objective includes:

- Analysis on the three types of deformation for the two structures using COMSOL Multiphysics
- Compare the two models using the deformation analyses
- Verify the solutions from the two models using analytical exact solution
- Compare the time and memory taken to study the two models

2 LITRETURE

2.1 Stresses in 3D

Stress analysis is a typical study of physics and essential tool in the engineering field. It is a study of design of structures of different fields in mechanical part under a load. Deformation occurs when a force is applied on a body and internal forces are induced in the body. Stress are grouped into two fundamentals depending on the response exerted, they are called normal and surface stress (shear stress). The normal stress is also called tensile or compressive stresses. In Figure (1) and (2), The normal stress is acting perpendicular in opposite sides of the surfaces. The normal stress is expressed in x, y and z components and the units are force per unit area. Shear stress is the force per unit area that act perpendicular to the surface. In Figure (1c), Surface stress are acting in pair in two faces to form equilibrium. [8] [9]

In a condition, that stresses are acting on the same plane then the resultant is calculated with the addition of the vectors. In normal stress, the body is uniform cross-sectional where the stress is applied. The normal stress can be either tensile or compression depending on the direction. The formula for normal stress is given in equation and the unit is force per unit area [8]

$$\sigma = \frac{F}{A} \qquad (1)$$

Where

- F is the load applied and A is the cross-sectional area
- σ is the average normal stress
- A is the cross-sectional area

The stress that is acting in a tangent to the cross-sectional area is called the shear stress. The shear stress can be either negative or positive depending on the rotational direction (clockwise or anticlockwise). [8]

$$\tau_{avg} = \frac{V}{A} \qquad (2)$$

Where

- V is the shear force
- τ_{avg} is the average shear stress at a section
- A is the cross-sectional area

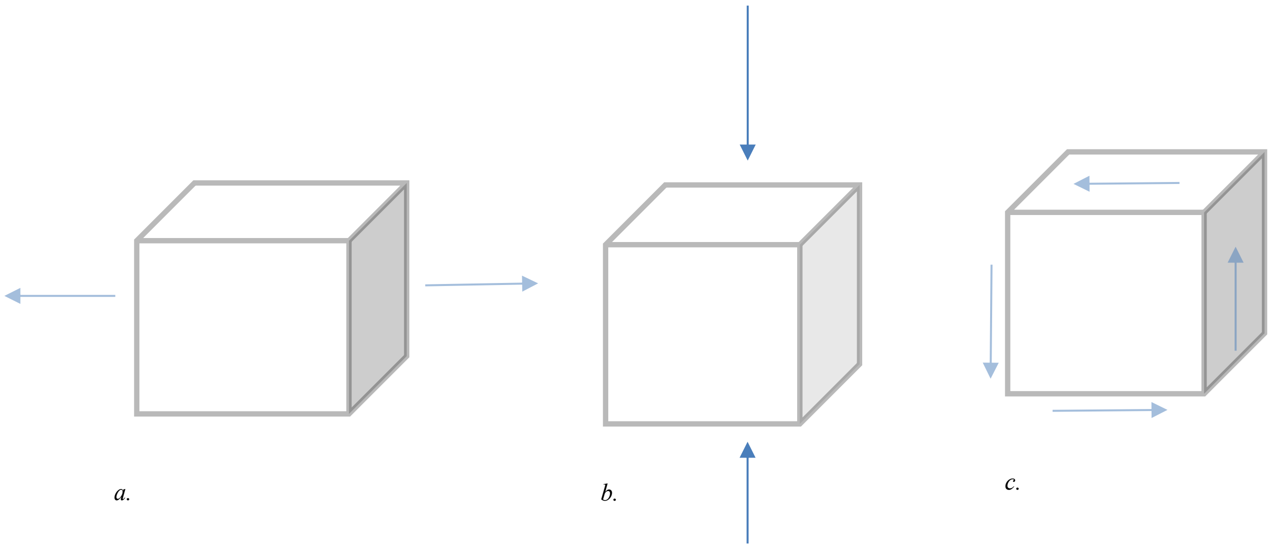


Figure 1 A body under a state of stresses a) Tension b) Compression c) Shear stress

The method of solving the shear stress is the same as the normal stress because the load is applied in uniform state. Uniform state of stress is where the object of the stress is uniform throughout. The force applied on a different plane can be the same but the stress on different planes will be different since the area of each planes are not equal. The stress is applied in four side of the surface to maintain equilibrium. [8] [9]

The state of stress in a cartesian coordinate at a point can be described using six components in Fig.2. The normal stress are the axial stress noted by σ_x, σ_y, σ_z. The shear stress is a load applied parallel to the area. The stress tensor is symmetric from the equilibrium equation. The stress tensor are reduced from nine components with 3 x 3 matrix to six components. The stress tensor denoted by σ_{ij}. The notation i specifies the face in which the stress acting. The notation j indicates the direction in which a stress is acting. The notation represents second order notation called stress tensors. [10]

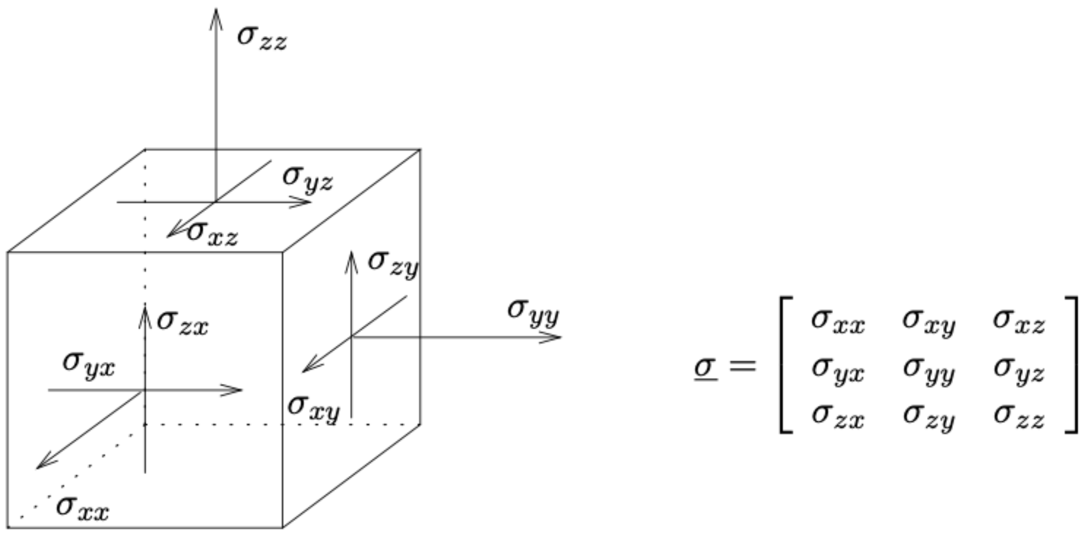


Figure 2 3D stress components [11]

Three stress in the x, y, z axis and six shear stress τ_{xy}, τ_{xz}, τ_{yz} are considered in Figure 4. Under Equilibrium or stationary position the shear is symmetrical along the axis acting having equal and opposite shear stress components. [8]

$$\sigma_{ij} = \sigma_{ji}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad (3)$$

Where

- σ_x, σ_y, σ_z is a stress in x, y, z
- τ_{xz} is the shear stress of x in the direction of z
- τ_{yz} is the shear stress of y in the direction of z
- τ_{xy} is the shear stress of x in the direction of y

2.2 Strain 3D

Strain is defined as the change in length per unit length shown in equation. A stress applied to a body tends to show change in length or undergo deformation. The materials shows a displacement field in any direction undergoing a deformation. Deformation not only cause the displacement in elongation and contraction but also deforms in direction called shear strain. Normal strain shows a change in volume while shear strain changes in shape or angle. [10]

In structural mechanics the displacement of a particles under deformation will be resolved under a components *u, v, w* parallel to the coordinates x, y, z. The general state of a strain at a point in a body is represented by three components of a normal strain is denoted using ε_x, ε_y, ε_z. The three components in the shear strain is represented using γ_{xy}, γ_{xz}, γ_{yz}.When the original length of a structure is dx and after deformation the length will be $dx + du$. After deformation the equation for the engineering strain will be [8] [12]

$$\varepsilon_x = \frac{(dx+du)-dx}{dx} = \frac{du}{dx} \quad (4)$$

The strain- displacement relation for a deformation of a beam are. [10]

$$\varepsilon_x = \frac{\partial u}{\partial x} \;;\quad \varepsilon_y = \frac{\partial u}{\partial y} \;;\quad \varepsilon_z = \frac{\partial u}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} ; \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x} ; \gamma_{yz} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right) \quad (5)$$

The strain tensor for 3-dimensional deformation object in an elastic body is described by nine components presented by equation 6. The notations are used to identify the Cartesian coordinate. The first notation identifies the plane in which a strain is acting. The second notation identifies the direction of the strain. [10]

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} \quad (6)$$

2.3 Stress-Strain relations

The stress-strain relations are described by using Hook's law in an equation given in table(1). The relation or proportionality of a stress and strain is called Hook's law. Which combines the stress and strain to give a stiffness relation. Poison ratio is used for analyzing more than one dimensional relation between all direction. In two dimensional, the material shows isotropic behavior and is pulled in two direction having a poison ratio being a factor. [8]

The normal strain of 2 dimensional structure in both direction x and y is given by: [10]

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\, \sigma_y \right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\, \sigma_x \right] \qquad (7)$$

In 3D dimensional the Hooke's law is applied similarly as one dimensional and two-dimensional bodies. The equation of equilibrium is applied in linear elastic deformation. In one dimensional stress- strain, one component is only considered. In the case 2 and 3 dimension two and three components are considered. [10] [9]

The relation between the stress and strain are expressed by using parameters called Young's modules and Poisson ratio. the Poisson ratio is unitless, but Young's modules is represented by force per length square. The two parameters are useful to create a relation in different direction that is the extended due to the stress exerted. [10] [9]

Table 1 Equations based on Hook's Law

Hooks Law in Cartesian Coordinates.	
$\sigma_x = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\,\varepsilon_x$	$\varepsilon_x = \frac{1}{E}[\sigma_x - v(\sigma_y + \sigma_z)]$
$\sigma_y = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\,\varepsilon_y$	$\varepsilon_y = \frac{1}{E}[\sigma_y - v(\sigma_x + \sigma_z)]$
$\sigma_z = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\,\varepsilon_z$	$\varepsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_x + \sigma_y)]$
$\tau_{xy} = 2G\,\varepsilon_{xy}$	$\gamma_{xy} = \frac{2(1+v)}{E}\tau_{xy}$
$\tau_{yz} = 2G\,\varepsilon_{yz}$	$\gamma_{xz} = \frac{2(1+v)}{E}\tau_{xz}$
$\tau_{zx} = 2G\,\varepsilon_{zx}$	$\gamma_{yz} = \frac{2(1+v)}{E}\tau_{yz}$

- Where λ and G are the lame's constants or parameters
- v is the Poisson's ratio and τ is shear stress

For isotropic materials in table 1, the relationships are expressed using Poisson's ratio's and young modules. [10] [9]

$$G = \frac{E}{2(1+v)} \quad (8)$$

$$, \lambda = \frac{Ev}{(1-2v)(1+v)} \quad (9)$$

- Where E is the young modules, G is the shear modules, v is the Poisson's ratio and τ is shear stress

2.4 Linear elasticity

Elasticity is defined as the ability of a body to resist the influence of the stress. The body gains the shape of the body after removal of the stress. The influence of external force is also affected with the geometry and mechanical properties of the material. Elasticity is directly related with the distribution of stress, strain and displacement in case of elastic material under the influence of forces exerted. If the force acting does not exceed the limit the deformation will be reversed to its normal state. [13]

Linear elasticity is a general three-dimensional theory based on Cauchy's work which was discovered in 1820's. The linear elasticity is a model in which the stress is directly related with the strain which is called Hook's law. For example, when a direct force is applied on an object there is a change in the strain which is deformation. Linear elasticity is characterized using Poisson ratio and young's modules. The relation between the transverse and axial direction in a strain is defined by a Poisson's ratio. [6] [13]

$$v = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} \quad (10)$$

Where [14]

- ε_{yy} is the transverse strain
- ε_{xx} is the longitude strain
- v is the Poisson's ratio

2.5 Equilibrium Equation

The equilibrium of a structure requires a balance of force and moment to stop the body from translating in any direction. The force and moments can be solved into components along x, y, z coordinates. The analysis of the loadings is done using solid mechanics equation by analyzing as it elongates, rotates and bends. [13]

Equation of equilibrium [15]

$$\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + F_x = 0$$

$$\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + F_y = 0$$

$$\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right) + F_z = 0 \quad (11)$$

The force of the body, F_x, F_y, F_z is per unit of volume in the x, y , z direction and represents gravitational forces. The equation in a large displacements should be in a deformed position. The stresses are defined as force per unit of deformed area. [15]

2.5.1 Uniaxial Loading (Elongation)

The uniaxial loading is defined as a stress as a specific force that is force divided by uniformly distributed along the cross – sectional area. The behavior of the material also depends on the type of stress and strain used. Hook's law is applied when the materials is behaving in a linear elastic and homogeneous materials. Gradually the cross- sectional area changes with the concentration of load that is applied. The deformation occurrence is directly related to the stress applied. There are different kind of factors affecting the uniform distributed load could be the weight of the bar or friction forces along the surface. The tensile force is considered as positive and the compressive force as negative. [16]

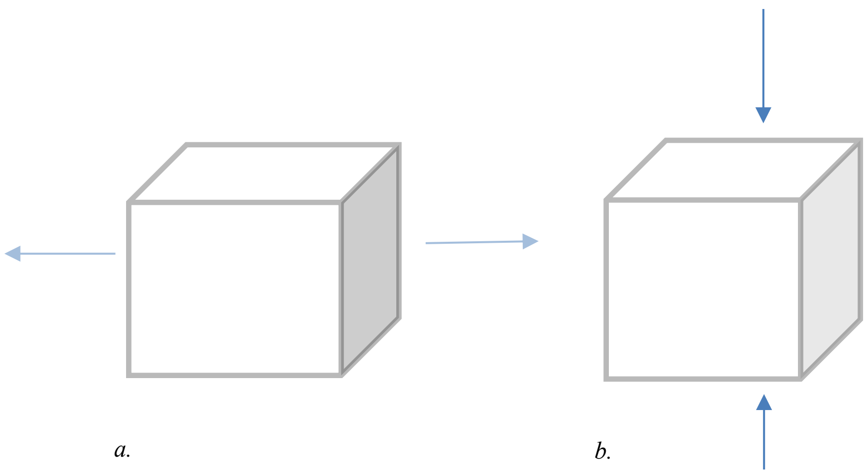


Figure 3 a) discribe the tension b) shows a compression of an object

The area is considered as a function of its position when the area across the bar varies. The loads are in different direction and considered positive referred to as elongation in figure(3a). The compressive are negative in the same direction are called contraction in figure(3b). The displacement of the axial loading occurring from one end to another is calculate by relating the internal load with the stress and relating displacement to the strain. The equation can be defined combining the hooks law to determine the displacement occurred during actual axial loading. [16]

$$\delta = \frac{PL}{AE} \qquad (12)\ [17]$$

Where [16] [18]

δ = displacement from different ends

L = original length

A = -sectional area

E = modules elasticisty

P = force acting on a plane

A material in which a stress is exerted on the x direction will experience a strain in the x direction ε_x. A stress exerted in the y and z direction will also affect and encounter a strain ε_x defined in the Poisson's ratio. Similarly, for the strain in y direction the factors are stress in x and z direction. In the case of z direction, the stress exerted in both x and y direction affects the strain in z direction is given as [19] [17]

2.5.2 Torsion Deformation

Torque is a moment in the direction of about longitude axis that tends to twist. The torsion of non-circular cylinders was discovered by mathematician barre de Saint-Venant in 1850's. The discovery was explained the importance of wrapping displacement in the direction parallel to the angle of twisting. The deformation of an angle can be calculated when a solid body is subjected to a torsional load from one side while the other side is in constraint shown in figure(4). The differences in the angle on the z axis can be calculated as: [20] [17]

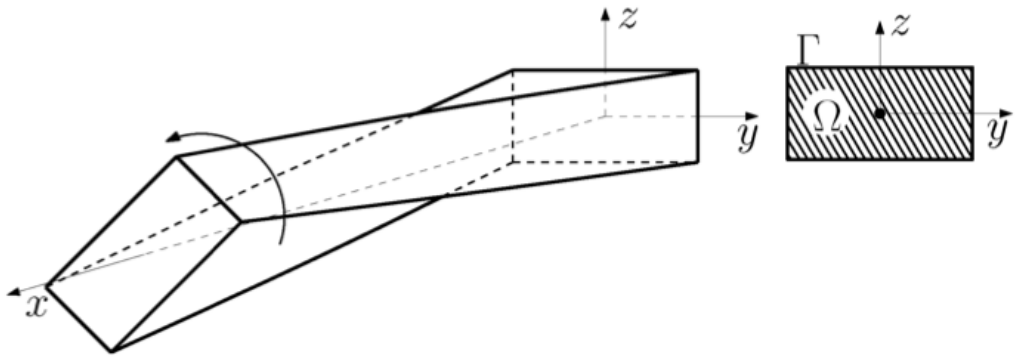


Figure 4 A figure of a body under Torsion [21]

The properties of the material of a cross-section where the torque is applied is called polar moment of inertia. For a solid circular shaft geometry, polar moment of inertia is given as [22]

$$J = \frac{\pi d^4}{32} \quad (13)$$

For a small angle, when a torque is applied to the solid circular cross-section geometry the shear strain is given as

$$\gamma = \tan(\theta) \quad (14)\ [22]$$

Where

- θ is a constant equal to the twist per unit length
- γ is the shear strain

When the radius of the solid circular cross-section geometry or shaft remains the same after a torque is applied the shear strain is calculated as

$$\gamma = \frac{\rho\theta}{L} \quad (15)$$

Where [22]

- γ is the shear strain
- θ is the rotational angle
- ρ is radius from the center of the shaft to the point the torque is applied
- J is polar moment of area

the torsional angle of twist of a the solid circular cross-section geometry is defined as : [3] [22]

$$\theta = \frac{TL}{GJ} \qquad (16)$$

Where [4]

- J is polar moment of area
- M is the moment
- L is the length
- T is the torque

2.5.3 Bending Deformation

The bending moment is a direct influence of external stress that tends to change or bend the body about an axis within the plane of area. Bending deformation is the change in the shape of elastic line due to the bending moment. Deformation in bending result in the rotation and displacement field. Consider a cross-sectional area of a bar in Figure (5). The amount of bending moment is directly proportional to the bending deformation. When the one in the bottom is stretching the top of the surface on the other ends seems to be compressing. In a case of deformation in figure (5), the horizontal lines are showing a curve, but the vertical lines show a straight line. The plane remains the same before and after deformation. [14] [17]

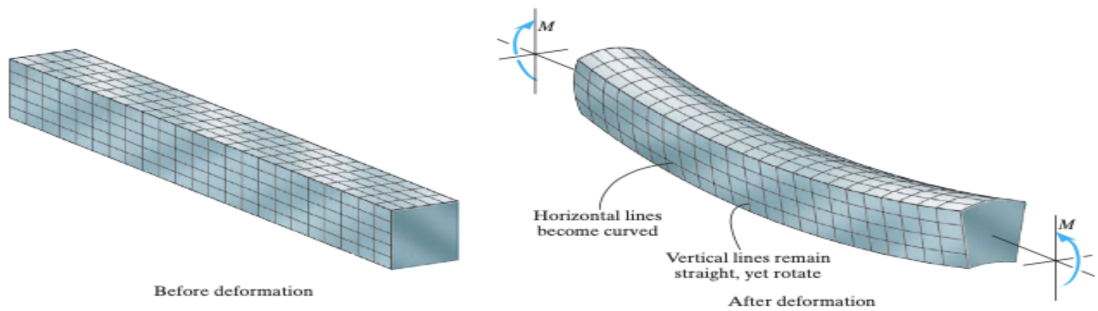


Figure 5 Before deformation and after [17]

Where the stress of a bending moment a linear elastic beam is [14] [22]

$$\sigma = \frac{Mc}{I} \qquad (17)$$

- M is the bending moment

- c is the distance between applied load and neutral axis
- I is the inertia on the neutral axis

In the D. Bernoulli's theorem of beam bending, the relation of the beam deflection and load is described. The equation satisfies the Hook's law of equation of equilibrium. [22]

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) = q(x)$$

$$v = \frac{PL^3}{48EI} \qquad (18)$$

Where

- *q(x)* is the load per unit length
- *v(x)* describes the maximum deflection in a rectangular cross-section beam
- *E* is the elastic modules and I is the moment of inertia
- I is the area moment of inertia and is given as $I = \frac{b*h^3}{12}$ [22]

2.6 COMSOL Multiphysics Software

COMSOL Multiphysics is a powerful software environment used in modeling and analysis in engineering, industries and research purpose. COMSOL Multiphysics software was founded by CEO Dr.h.c Svante Littmarck and President Farhad Saeidi in 1986. The first release was in 1998 with add-on module for structural mechanics, fluid flow, heat transfer, electromagnet and more. The software had extra features which allows software's like CAD, MATLAB and Excel to be integrated boosting its functionality. [3]

COMSOL Multiphysics has a powerful GUI to try and explore different types of models. A predefined simulation platform that is used for multipurpose is based on partial differential equations (PDE). Solving the PDE takes a time to equate the given mathematical equations, properties and boundary conditions. Computing in COMSOL Multiphysics doesn't need deep knowledge in mathematics or physics due to the built-in physics modes with a setup equations and variables. The software allows the user to predefine the model formulation and analysis with flexibility and adoptability defined with PDE. COMSOL Multiphysics uses several applications and the most widely used are: [3]

- Acoustics Module
- CAD Import Module
- AC/DC Module
- Chemical Engineering Module
- Earth Science Module
- Heat Transfer Module
- Material Library
- RF Module
- Structural Mechanics Module

The structural mechanics is a solution for the field of structure and solid mechanics. The structural mechanics has contributed to the world of engineers, researchers and teachers. The tools help the user to learn with a short period of time. The interface encompasses of multi Multiphysics models and methods. [4] [3] [5]

The solid objects show a deformation characteristic due to the force and constraint applied. During the deformation the spatial coordinates changes while the materials coordinates is kept steady.

Some of the method analysis are:

- Linear Elastic Materials
- Linear Viscoelastic Materials
- Piezoelectric Material
- Magneto strictive Materials
- Large deformation
- Contact

2.6.1 Solid Mechanics in COMSOL

The solid mechanics is a physics interface solving equations of a motion for solid materials. The model is used in structural analyses of 3D,2D or axisymmetric bodies. The interface shows the deformation of a solid object in 2 or 3-dimensional space. The coordinates are specified in lowercases x, y, z or r, φ, z in axisymmetric are called spatial coordinates. [1] The reference coordinates are used for materials particles denoted in uppercase variables X, Y, Z which are continuum mechanics theory. During deformation, when a force is applied on a solid body it tends to keep its coordinate vectors but in the case of spatial coordinates it changes with time and force in space. Both the special and reference coordinates are called global coordinate system in the physics interface. In the material frames, vectors and tensor are defined in the global coordinates for example solid. SXY is called the second Piola-Kirchhoff stress and solid.sxy is called spatial or Cauchy stress. Coordinate systems is used to describe for each material frames which is selected based on material added, load and constraint. [1] [5]

2.6.2 Beams Model in COMSOL

A beam is a slender component that is designed to support a load acting perpendicular to the longitude axis. A beam can be described by its cross section like area, moment of inertia and torsional constant. The end points of the beam can be free, supported or clamped. Beams have high tendency to sustain forces and moments in different directions. [1] [5]

The shape of the beam in axial deformation and twist around a beam shows a linear shape function. The bending moment and the corresponding rotation are showed by a function called Hermitian cubic function. [4] [5]

The Beam interface uses two- and three-dimensional frameworks of uniaxial beams. The components in the displacement, direction and rotational angle are: [1] [5]

- *u, v, w (*Global displacement)
- $\theta_x,\ \theta_y, \theta_z$(rotational angle along the global displacement)
- x, y, z (displacement in the local direction)

The Beam interface/cross-section uses either Timoshenko or Euler-Bernoulli theory when analyzing stress and strain. Strain-Displacement on the axial strain in 3 dimensional is dependent on the shape function and transversal coordinates. [1] [5]

$$\varepsilon = z_1 \frac{\partial \theta_{1y}}{\partial s} - z_1 \frac{\partial \theta_{1z}}{\partial s} + \frac{\partial \mu_{axi}}{\partial s} \qquad (19)$$

Where

- z_1 and y_1 are the transversal coordinates
- $\frac{\partial \theta_{1y}}{\partial s}$ is the tangential derivative along the edge of a rotational angle in the direction of y
- $\frac{\partial \theta_{1z}}{\partial s}$ is the tangential derivative along the edge of a rotational angle in the direction of z
- $\frac{\partial \mu_{axi}}{\partial s}$ is the tangential derivative of the axial displacement in the edge direction

Stress-Strain Relation The axial deformation for stress – strain is denoted by [5]

$$\sigma = E\varepsilon_{el} + \sigma_i \quad (20)$$

Where

- σ_i is the initial stress

The stress-strain relation for the torsional and shear deformation is

$$G = \frac{E}{2(1+v)} \qquad (21)$$

Where [5]

- $E \ is \ the \ young \ modulas, \sigma_i$ is the initial stress
- v is the Poisson ratio, G is the shear modules, $\gamma \ is \ the \ shear \ strain$

A beam is structured which can be expressed by using beam cross section. In beam cross sectional structure, it is possible to analyze both 2D and 3D. The structure interface provides the best outcome to describe data. The interface exceeds the allowed values of the stress. The interface is based on the cross-sectional properties not the true values.. The normal stress is calculated as [5]

$$\sigma_n = \frac{N}{A} \qquad (22)$$

The normal stress based on bending moment is the same in equation(17) and calculated as [5]

$$\sigma_{bk} = \frac{M_{1y} z_{1k}}{I_{yy}} - \frac{M_{1y} y_{1k}}{I_{zz}} \qquad (23)$$

Where

- M_{1y} is the initial bending moment
- I_{yy} and I_{zz} is the moment of inertia in respective directions

In 3D the M_{1x} is used in a torsional stiffness of a beam and moment is defined as [5]

$$M_{1x} = GJ\left[\left(\frac{\partial\theta_{1x}}{\partial s} - \left(\frac{\partial\theta_{1x}}{\partial s}\right)_i\right)\right] + M_{ix} \quad (24)$$

Where

- M_{1x} is the torsional moment
- M_{ix} is the initial torsional moment

The result for maximum shear stress due to torsion is available in 3 dimensional and is calculate as [5]

$$\tau_{t,max} = \frac{|M_{1x}|}{W_t} \quad (25)$$

Where

- W_t is the torsional modulus
- M_{1x} is the moment at x

3 METHOD

The software supports pre-existing/ built-in physics like structural mechanics. The method of analyzing two different kinds of mechanics models was conducted using COMSOL Multiphysics. The two-models analyzed are solid mechanics and beam model. The axial, bending and torsional deformation of the 3D model are analyzed. A block geometry is used in analyses for bending and axial deformation. A cylindrical geometry is designed to be able to compare the two models in torsion deformation. In the method section, the two structure model's and deformation are explained step by step. The software used to analyze the experiment is COMSOL Multiphysics version 5.5. The first step is choosing the type of physics to be used in order to model a three-dimensional model.

There are different kinds of models that can be studied. Thereafter, the type of study is chosen to be Stationary.

3.1 Parameters

A parameter is a static variable used to define a geometry. The parameters are useful to quickly adopt to a different configuration. The parameters that are considered in the 3D solid (solid mechanics) and beam model are the same in order to compare both models. The parameters are set for both of the interference or model depending on the type of deformation . There are three types of deformation that is considered, elongation, bending and torsion. When analyzing the model, deformation stress and strain are only considered, and stationary study is used because the case study was not time dependent. The parameters set are length, thickness, stress, torque, moment and force.

Table 2 predefined parameters used to recall

Name	**Expression**	**Value**
Length(le)	100[mm]	0.1 m
Thickness(th)	10[mm]	0.01 m
Force(Fn)	10[kN]	10000 N
Torque(Fed)	500000[N/m]	5E5 N/m
Moment(Fm)	5000[N*m]	5000 N·m

The geometry was set as a block for solid mechanics in axial and bending deformations. The geometry was set based on table two parameters having a width, depth and height.

Table 3 Geometry parameters used for a block in 3D model

Description	**Value**
Width	le
Depth	th
Height	th

For the case of torsion, the structure in the geometry set to be a cylindrical geometry. The geometry is set based on the same parameters as table 4 but with a set of radius 5 mm shown in table

Table 4.Geometry parameter used for cylindrical in 3D model

Description	**Value**
Radius	th/2
Height	le

Beam model geometry is created on the edge or point boundary's conditions. The beam cross-section is set with user defined common section which is rectangular cross-section seen in table (5). In the case of Beam model, the geometries are made with line segment coordinates in x with length of 0.1m in bending and axial deformation. The Euler-Bernoulli theorem is used to analyze and study the material properties.

Table 5 parameters used in the line for Beam model

Description	**Value**
Specify	Coordinates
Coordinates	{0, 0, 0}
Specify	Coordinates
Coordinates	{le, 0, 0}
Cross section definition	Common sections
Section type	Rectangle
Width in local y direction	th
Width in local z direction	th
Orientation vector defining local y direction, X component	0
Orientation vector defining local y direction, Y component	0
Orientation vector defining local y direction, Z component	1
Rotation of vector around beam axis	0

For torsion deformation, the beam cross-sectional is set with a user defined cylindrical cross-section. The diameter is set with a value of thickness that is the 10[mm]. The coordinate and orientation of the vector used in a cylindrical geometry is the same in the block give in table (5).

Table 6 circular cross-sections in torsion in beam model

Description	Value
Cross section definition	Common sections
Section type	Circular
Diameter	10[mm]

The parameters are set to the geometries in the two models having a shape of a cube, cylindrical and a line shown in figure (6). The cubic shape geometry is used for the bending and axial deformation analysis in 3D solid. The cylindrical shape geometry is used in the torsion deformation analysis in 3D solid. In the beam model, the geometry used in the analysis are the same having a different cross-sectional input specified in table (5) and table (6).

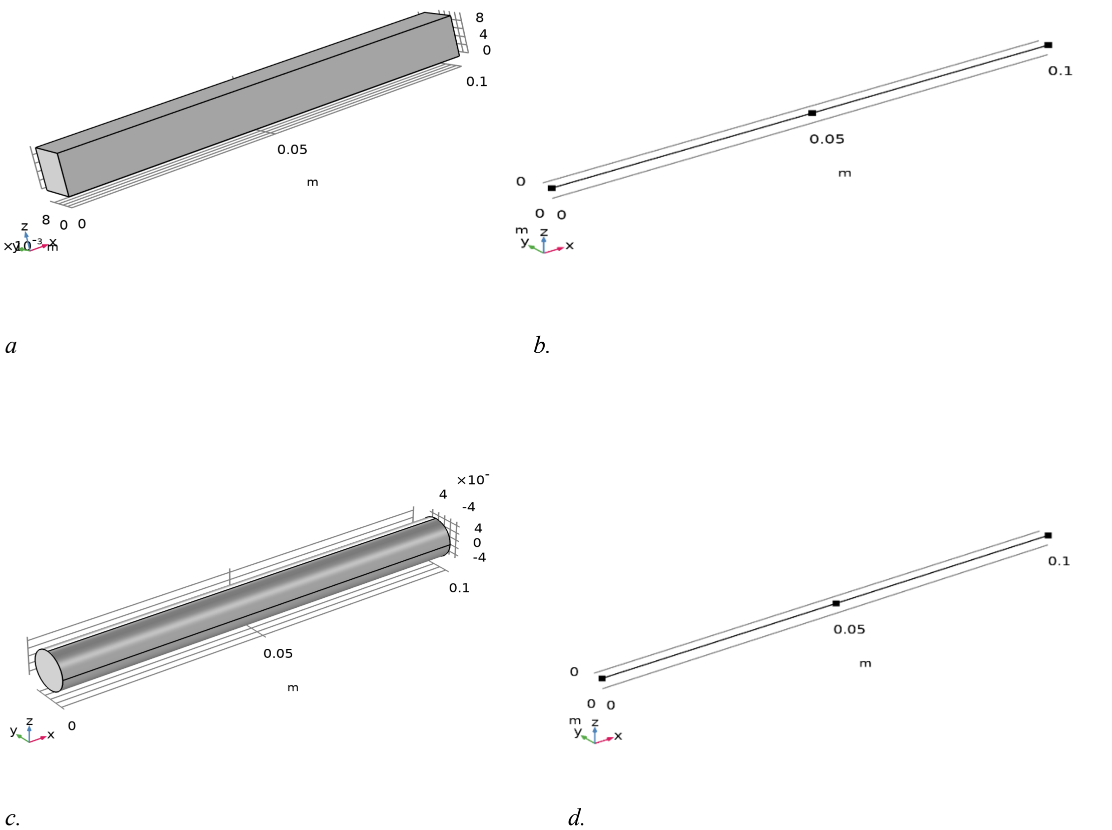


Figure 6 The geometry that is build using a COMSOL Multiphysics for deformation in axial bending and torsion b) Geometry of the block for 3D solid b) Geometry for the line for beam model c) Geometry of the cylinder for 3D solid d) Geometry for the line for beam model

The next step is to choose the materials that is going to be used for both the models. Both the models are using the Steel ASI 4340 having a predefined material properties.

Table 7 Materials Properties of Steel ASI 4340

Name	**Value**	**Unit**
Density	7850[kg/m^3]	kg/m³
Young's modulus	205e9[Pa]	Pa
Poisson's ratio	0.28	1

3.2 Boundary conditions

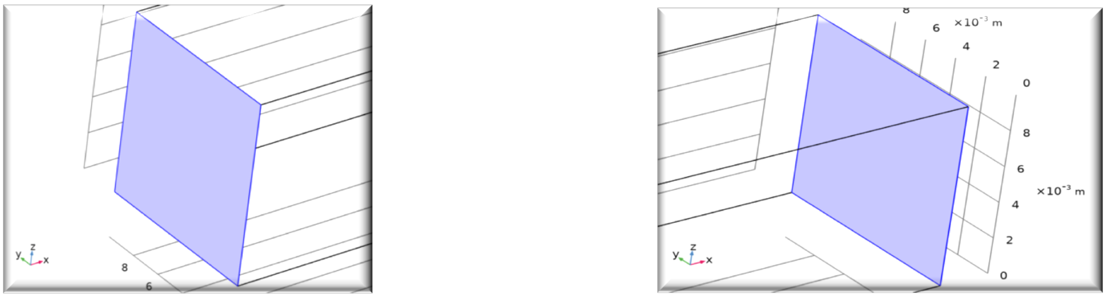
3.2.1 Case Solid Mechanics Modeling

Uniaxial loading

The coordinate and boundary conditions is given based on the method of the analysis. In case of uniaxial elongation, the boundary load is added in the x axis in both directions in figure(7a) and figure(7b). The boundary load is set with in the positive and negative co-ordinate of x with a stress of 100[MPa] shown in figure 8.

Table 8 boundary conditions in uniaxial loading 3D solid

Description	**Value**
Load type	Force per unit area
Load	User defined
Load in the left[MPa]	{-100, 0, 0}
Load in the right[MPa]	{100, 0, 0}



a. b. .

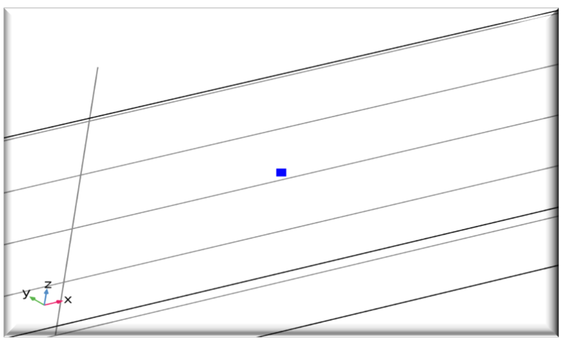
Figure 7. stress applied in both directions a) shows the stress applied in -x direction b) shows the stress applied in x direction.

Bending:

when setting the boundary condition, the block is fixed by putting boundary constraint in both side of the edges in figure (8a) and figure (8b). A point coordinate is created along the middle of the block to specify the position of the load shown in a table(9). The point load was set perpendicular to the longitudinal direction of z axis. The load is specified manually in the direction of negative z coordinate that is 10[kN].

Table 9 boundary conditions given in bending 3D solid

Description	Value
Point coordinate	{0.05, 0.005, 0.01}
Load type	Total force
Point load	User defined
Point load	{0, 0, -Fn}



a.

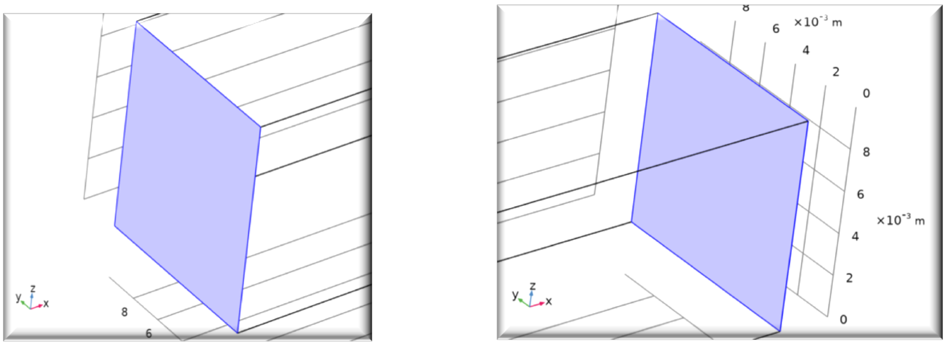


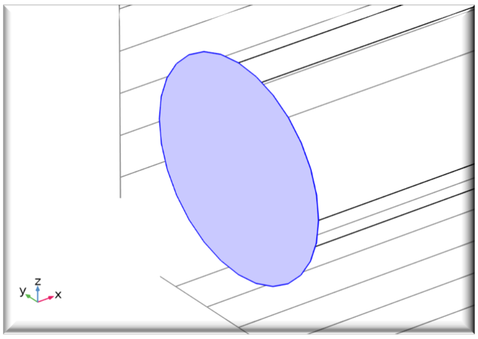
Figure 8 Bending structure of a body a) point load at a midpoint b)fixed constraint at the edge c) fixed constraint at the edge

Torsion

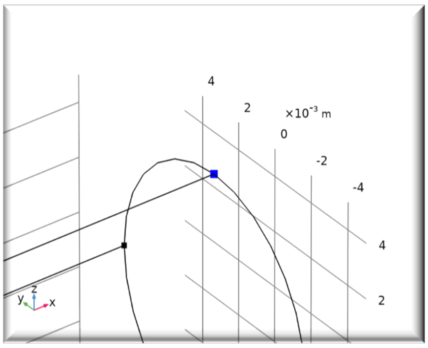
when setting the boundary condition, the cylindrical geometry is given a fixed constraint in the left edge shown in figure (9a). The other edge, putting a two coupling point loads at right edge is considered in figure (9b) and figure (9c) . The point loads is given on top and bottom of the cylinder in different direction to create a twist. The point loads added will create a torsion in a clockwise direction. The point load is added in the y direction given in table (10). The two direction point loads in one end and constraint in the other end will result in a twist deformation.

Table 10 boundary conditions for a torsion in 3D model

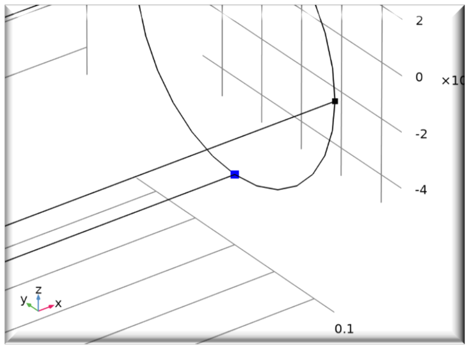
Description	**Value**
Load type	Force per unit length
Load	User defined
Load Point Edge 2	{0, Torque, 0}
Load Point Edge 3	{0, - Torque, 0}



a.



b.



c.

Figure 9 boundary loads and fixed constraint in two ends of the edges a) fixed constraint in the edge b) point load in the direction of the other edge in y c) point load in the direction of the edge in negative y

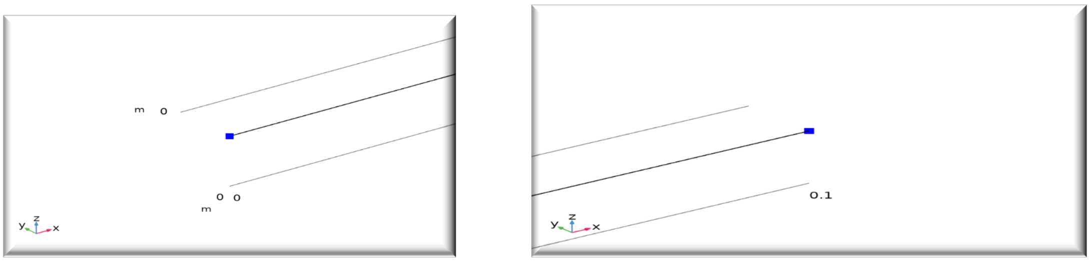
3.2.2 Case of Beam Modeling

Uniaxial loading

The point load is added both edges long the x axis. uniaxial stretching, the boundary load is added in the positive and negative coordinate of x with a stress of 10[kN] shown in figure (10a) and figure (10b).The values are added in the boundary conditions in table(11).

Table 11 boundary conditions for a uniaxial loading in Beam model

Description	Value
Point load	User defined
Point load 1	{-Fn, 0, 0}
Point load 2	{Fn, 0, 0}



a. *b.*

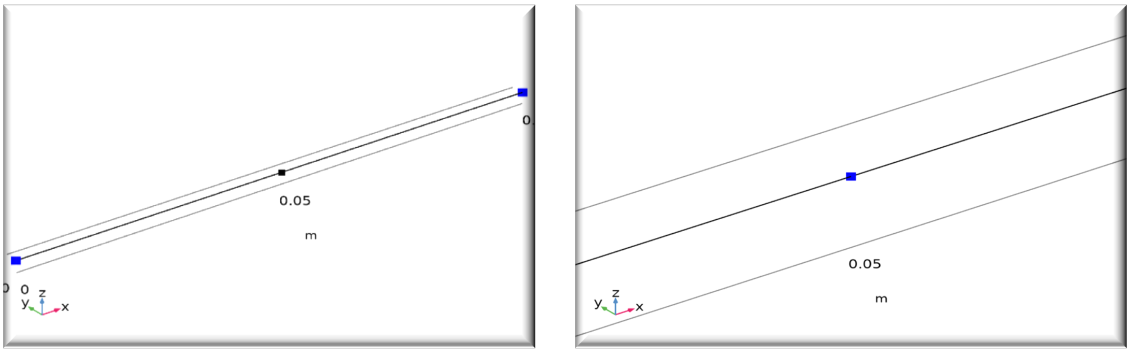
Figure 10. a) point load in the negative direction b) point load in the positive direction

Bending

when setting the boundary condition, the block is fixed by putting point constraint in both side of the edges in figure(11a). A point in the middle of the beam is constructed between the two edges shown in figure(11b). The point is plotted to specify the coordinate of the load applied. The load point was set in the longitude direction which is the z axis of -10[kN] in table(12).

Table 12 boundary conditions for a bending in Beam model

Description	Value
Point load	User defined
Point load	{0, 0, -Fn}



a. *b.*

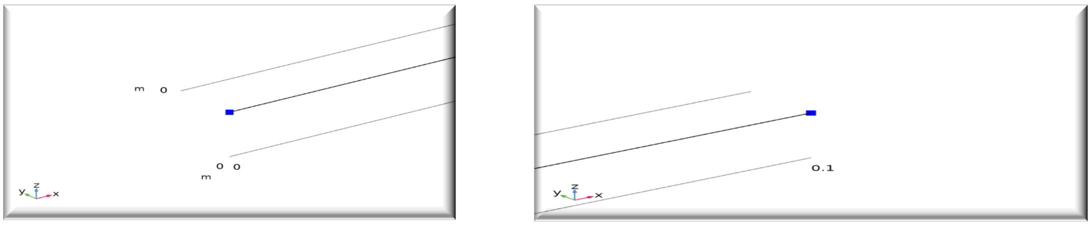
Figure 11 a) fixed constraint in both ends of the edge b)point load in the negative z direction

Torsion

when setting the boundary condition, the line in component 1 have a fixed constraint in figure (12a). The other end point are given moment in the x component in Table 11. The moment of the edge points in figure (12b) is placed with a 5000 N·m. When the one end is fixed and the other edge point is given a moment giving a twist deformation.

Table 13 boundary conditions for Torsion in Beam model

Description	**Value**
Point load	User defined
Point load 1	{0, 0, 0}
Point load 2	{0, 0, 0}
Point moment, x component 1	0
Point moment, y &z component	0
Point moment, x component 2	5000 N·m
Point moment, y &z component 2	0

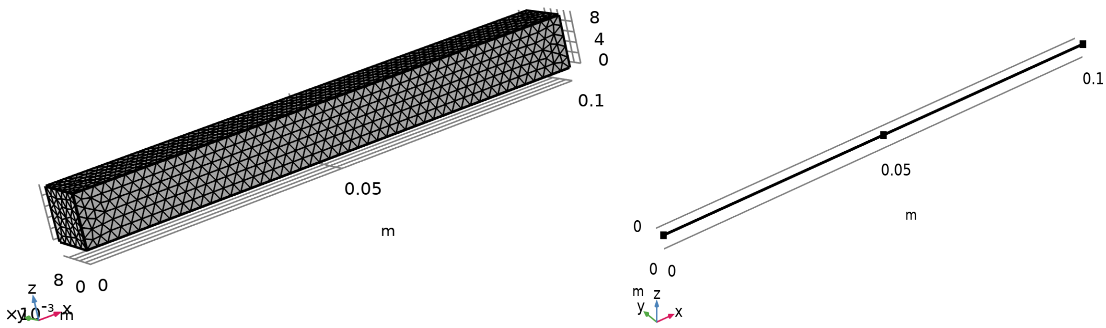


a. *b.*

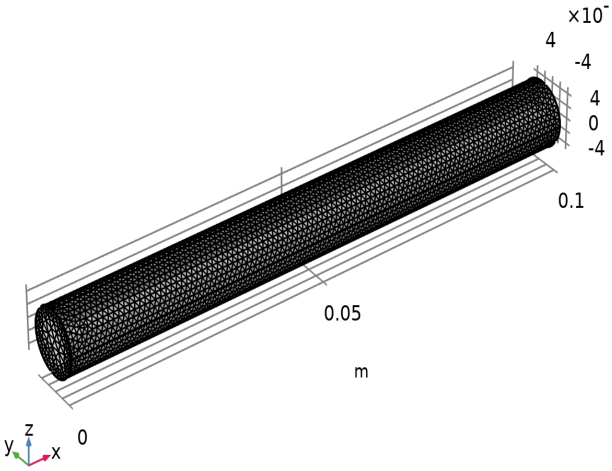
Figure 12. positive and negative moment is added in left and right points. a) Fixed constraint point moment 1 b) point momentum 2 in the x component

3.3 Mesh

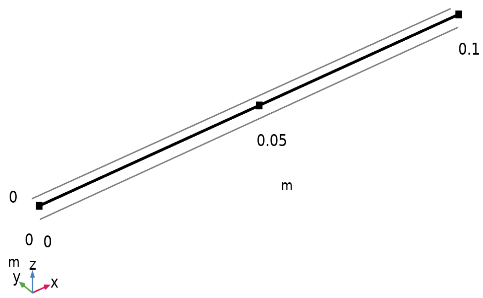
COMSOL provides the user with flexible and easy access solving environment. The default shape used to mesh the material is tetrahedral element. The size of the mesh selected is to be extremely fine meshing. The extremely fine meshing is found under the physics-controlled mesh. The tetrahedral mesh has a shape of a triangular structure used in two and three-dimensional objects. In figure 13, the tetrahedral mesh is observed in both the solid mechanics and beam models.



a. *b.*



b.



d.

Figure 13. Mesh for a cubic, cylindrical and line geometry in 3D model and Beam model a) Extremely fine Mesh in 3D solid b) Extremely fine Mesh in Beam model c) Extremely fine Cylindrical Mesh in 3D solid d) Extremely fine line Mesh in Bam model

3.4 Analytical Approach

The study of the analysis is based on three deformations, bending, torsion and axial elongation. The theoretical analysis will be calculated based on the equations that are given in the literature. The theoretical analysis are useful to validate and used to compare the results used for the experiment. Mathematical analysis is conducted in the result for the three deformations using three equations.

Case1. The Uniaxial displacement formula given in equation (12)
Case2. The bending deformation displacement is given in equation(18) where I is calculated for a rectangular cross-section.
Case 3. The torsion deformation displacement angle is given in equation (16) where J and G is calculated using equation (13) and equation (8).The shear strain formula stated in equation(15) is used to change shear strain to rotational angle for 3D solid.

4 RESULT

4.1 Theoretical Analysis

The study of the analysis is based on three deformations, bending, torsion and axial elongation. The theoretical analysis will be calculated based on the equations that are given in the literature.

Given: P= 10(kN),L=0.1m,E=205(GPa),A= 0.0001m^2

Case1 : The displacement of the axial deformation is given as from equation

$$\delta = \frac{10kN * 0.1m}{0.0001m^2 * 205GPa} = 0.00004878049 \text{ m} = 4.878 * 10^{-5}m$$

Case 2: The bending deformation for a rectangular cross-sectional geometry is given by displacement

Given:

$$v = \frac{10kN * (0.1)^3}{48 * 205GPa * \frac{0.01 * (0.01)^3}{12}} = 0.0012195122\ m$$

Case 3: The torsional deformation for a rectangular cross section is given as

$$\theta = \frac{5000N.m * 0.1m}{80.07E9\frac{N}{m^2} * 0.09817 * (0.01)^4} = 6.3609\ rad$$

Table 14 Values calculated using the formula given in the literature

Description	**Value**
Axial displacement(m)	4.878*10^{-5}
Bending displacement(m)	1.219512*10^{-3}
Torsion(rad)	6.3609

4.2 COMSOL Multiphysics Analysis

The results are as follows based on the simulation in COMSOL Multiphysics. A three-dimensional model is built and solved in solid mechanics and Beam model with the given geometry parameters . The geometry of a solid mechanics is built with a block geometry and cylindrical geometry. The geometry for a beam model is built using a line with a rectangular cross-section and circular cross-section. The simulation is performed in a stationary analysis and using a default solver setting. An extra finer mesh physics-built system was built using tetrahedral element. The mesh result shows a global element size. The result will be compared based on deformation, time and physical memory. The result is shown below using a tables of the result gathered from the COMSOL Multiphysics. The result of the COMSOL analyses compares both the 3d model and beam model based on computed time, displacement and memory. The result is a verification of the validity between the 3D solid and beam interface models.

4.2.1 Uniaxial deformation in 3D solid

For axial deformation in 3D solid, the result is given in displacement in the x direction in table 15. The computed time and physical memory to perform study of the analysis in stationary is given to be in table 15.

Table 15 Study computed from COMSOL Multiphysics in uniaxial deformation 3D solid

Description	**Value**
Computation time	11 s
Physical memory	1.74 GB
Displacement in x component	$4.19*10^{-5}m$

From Figure(14), the surface displacement is observed in both side of the edges to have two values in opposite direction. The left side of the edge shows a displacement of 1.28E-5m in the negative direction. The right side of the edge shows a 2.91E-5m of elongation. In Table 12, the value is added both side of the edges.

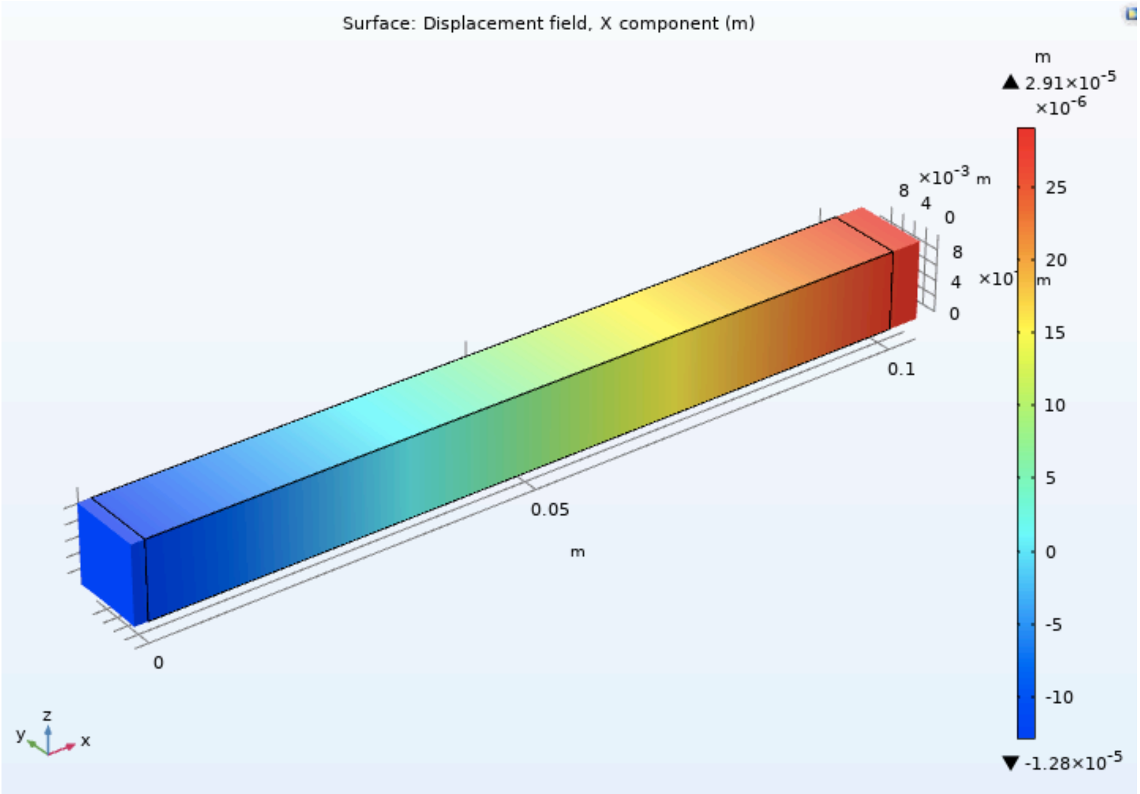


Figure 14 computation result in axial deformation displacement field in 3D model

4.2.2 Uniaxial deformation in Beam model

For axial deformation in beam model, the computation time and physical memory is stated in the Table 13. The computation time taken to perform the study is 1 second. The memory taken while performing a beam model is 1.35 GB. The normal strain is analyzed to equate the deformation /displacement in the x direction. The normal strain is found to be 4.8780E-5 from Table 13.

Table 16 Study computed from COMSOL Multiphysics in uniaxial deformation for beam model

Description	**Value**
Computation time	1 s
Physical memory	1.35 GB
Displacement in x component	$4.878*10^{-5}m$

The displacement values in x direction is observed in the edges are -4.3E-5m in the left side and 5.83E-6 in the right side in figure (15). When the values are combined, they are given in Table 13 to be 4.83E-5m.

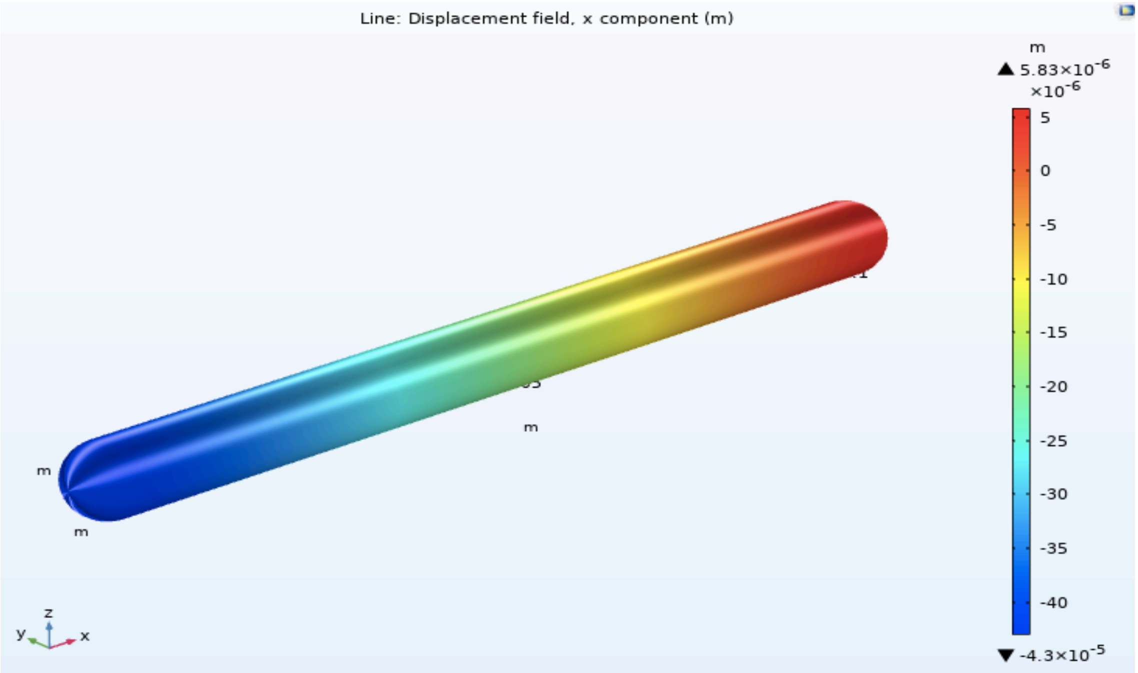


Figure 15 computation result in axial deformation displacement in beam model

4.2.3 Bending deformation in 3D Solid model

In the case of solid mechanics, the computational time that is taken to analyze the stationary study is 10 seconds. The physical memory taken to make the study is give in table 14. The displacement field in the z direction is negative because the deformation the load is applied downwards in the negative direction.

Table 17 Study computed from COMSOL Multiphysics in Bending for 3D solid

Description	**Value**
Computation time	10 s
Physical memory	1.32 GB
Displacement in x component	$1.3413*10^{-3}m$

The deformation is downwards in the middle at a point. The deformation displacement is only in the z direction. The displacement field a z direction is observed in the negative direction with a value of -1.34E-3 in figure(16).

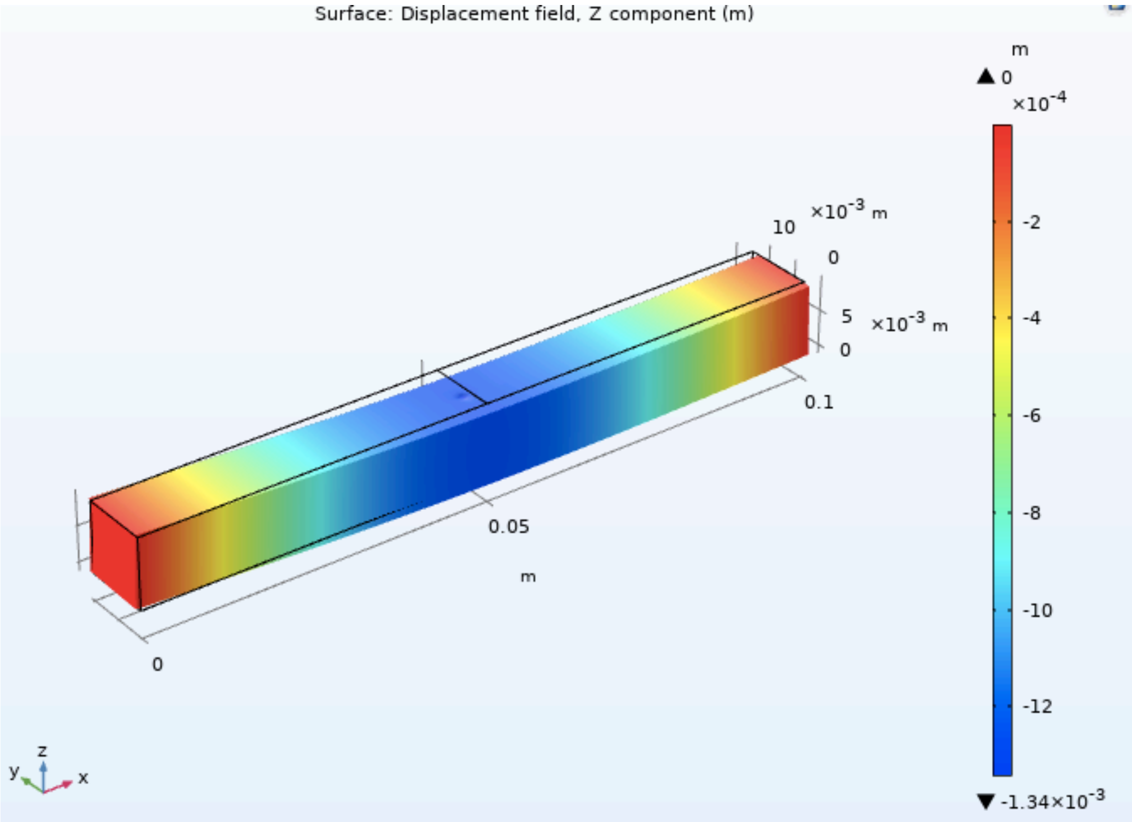


Figure 16 computation in bending deformation displacement in z in 3D model

4.2.4 Bending deformation in Beam model

In the case of beam model, the computational time that is taken to analyze the stationary study is 3 seconds. The physical memory taken to make the study is give in table 18. The displacement field in the z direction is negative because the deformation the load is applied downwards in the negative direction.

Table 18 Study computed from COMSOL Multiphysics bending in beam model

Description	**Value**
Computation time	3 s
Physical memory	0.98 GB
Displacement in x component	$1.2195 * 10^{-3} m$

The deformation displacement in the beam model is in the middle of z direction with a negative value observed in figure(17). The edges of both the ends are fixed while the load is applied in the middle.

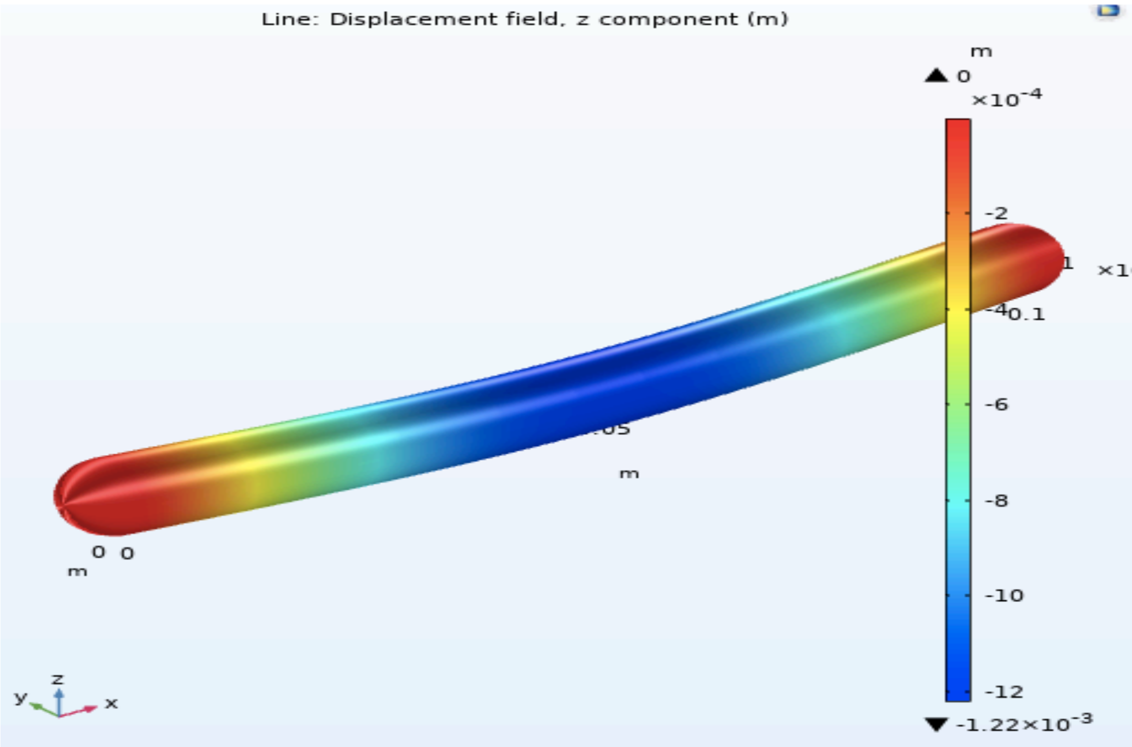


Figure 17 computation in bending deformation in beam model

4.2.5 Torsion deformation in 3D Solid model

In the case of torsion, the computational time to study the material was given to be 28 seconds. The physical memory taken to perform the study is 1.19 GB. A uniform load was coupled in all edges having a similar direction in the opposite side to create a torsion deformation. The torsion deformation is given in Table 19 to be a shear strain of

0.32411in the x direction. The rotational deformation angle of twist is calculate using a shear strain formula from equation 4.5. the result is calculated and shown in table 19.

Table 19 Study computed from COMSOL Multiphysics torsion for 3D solid

Description	**Value**
Computation time	28 s
Physical memory	1.19 GB
Shear strain in yz component	$1.2195*10^{-3}m$
Angle of twist in x axis	6.4822 rad

In figure(18), the surface analysis is done using the rotation of deformation in 3D solid model. The result in figure 19 shows the whole surface rotational deformation tensor in *yz* component(shear strain in *yz* component) at the end of the left edge have a bigger impact in deformation.

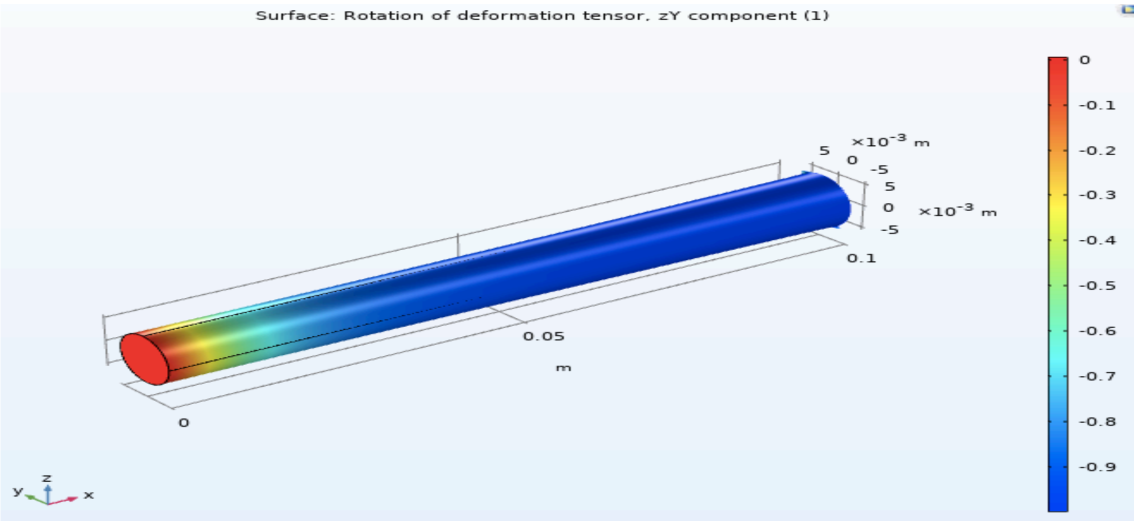


Figure 18 computation of the rotational deformation in 3D solid for torsion

4.2.6 Torsion deformation in Beam model

In this case, the computational time that was taken to analyze the geometry in beam is 3 seconds. The physical memory that is taken to perform the study is 1.31 GB. The maximum torsional shear strain was analyzed since the beam model does not encompass the rotational deformation strain. The value is in rad which is shown in Table 17.

Table 20 Study computed from COMSOL Multiphysics torsion in torsion for beam model

Description	Value
Computation time	4 s
Physical memory	1.11 GB
Angle of twist in x axis	6.3600 rad

In figure(19), the computation in the rotational field is done in the beam model. The result shows the rotational angle stated in the figure(19) and table 20.

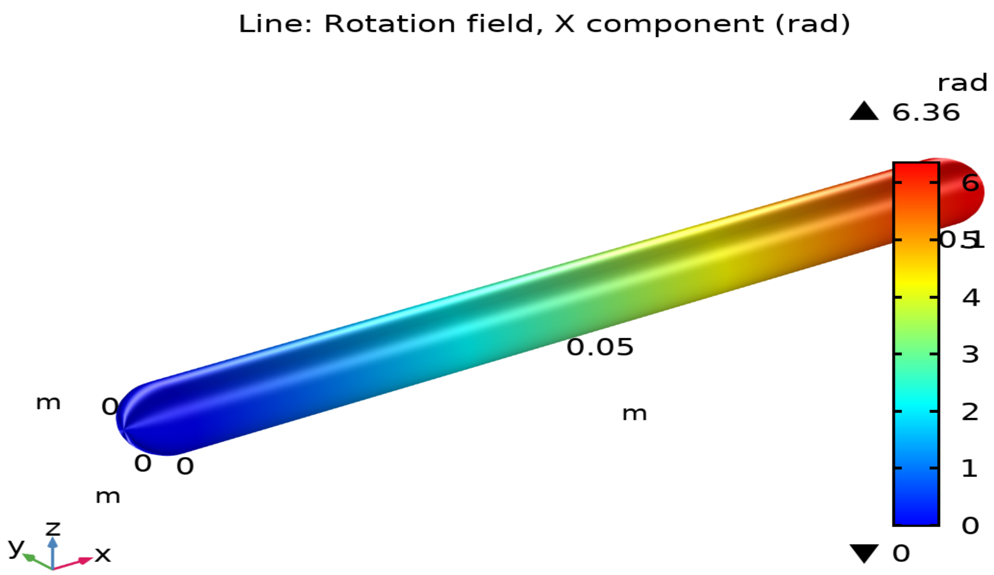


Figure 19 computation on rotational field in beam model for torsion

5 DISCUSSION

5.1 Comparison of Axial loading

In Table , it can be seen from the graph when a load is applied in both direction there is a stretch along the x axis. The displacement value that is found in both 3D model and beam model have different but closer values. The time differences that takes to study the geometry is given in table 18. In 3D solid modeling, the memory used when using tetrahedral mesh takes 11 s to complete one simple geometry. In the case of the beam model, it takes 1 s showing a better time. Showing a better computation time in beam model than 3D model. The displacement value in a 3D solid model interface shows a decrease in 14% compared to the analytical calculation. In beam model, the value shows an accurate value giving an accurate value with the analytical calculation. The simulation value shows a better computation time, memory and accurate value.

Table 21 comparison between the beam and 3Dsolid in axial loading

Description	**3D solid**	**Analytical calculation**	**Beam model**
Computation time	11s		4 s
Physical memory	1.74 GB		1.35 GB
Displacement	$4.19{*}10^{-5}m$	$4.878{*}10^{-5}m$	$4.878{*}10^{-5}m$

5.2 Comparison of Bending deformation

The point load is applied in the longitude(z) direction downwards using a force of 10 (kN) . The result from the displacement in the z direction in 3D solid model is -0. 0013413m.In Figure 19, the result shows a displacement of -0.0012195m in beam model. The result shows a negative values due to the direction of the force. The result shows 10% error in 3D solid model compare to analytical solution. In the Beam model, the values show 100% accuracy. The time taken to perform the study shows a 3 second in beam model and 18 second in 3D Solid model. The physical memory is observed in the beam model shows a decrease in 0.32 GB in beam model.

Table 22 comparison between the beam and 3Dsolid in bending

Description	**3D solid**	**Analytical calculation**	**Beam model**
Computation time	18s		3 s
Physical memory	1.32 GB		0.98 GB
Displacement	$1.3413*10^{-3}m$	$1.2195*10^{-3}m$	$1.2195*10^{-3}m$

5.3 Comparison of Torsion deformation

First, the values are evaluated in beam and 3D model are in shear strain . An equation is used to equate and change to angle of twist. The time that takes to make the study computation using 3D solid took 10 seconds. There is 7 seconds difference between the beam model and 3D solid making it more faster. The memory taken to evaluate the cylindrical geometry is a lot larger in 3D solid than the beam model. The values computed in COMSOL Multiphysics have close values with 3D solid and beam model seen in table 23. The value of the 3D solid have 1.92% error compared to the analytical solution. The error value in the beam model is close to zero with 0.014% difference. Therefore, the analytical solution show accuracy value with the beam model.

Table 23 comparison between the beam and 3Dsolid in torsion

Description	**3D solid**	**Analytical calculation**	**Beam model**
Computation time	10s		3 s
Physical memory	2.88 GB		1.03 GB
Rotational angle(rad)	6.4822 rad	6.3609 rad	6.3600 rad

6 CONCLUSION

The objective of the research was to investigate the analytical/theoretical analysis from the literature and make a base to compare both the beam model and 3D model. The analysis are made for axial, bending and torsion. A deformation displacement is used to compare for both axial and bending deformations. In a case of torsion, the rotational angle in radians is used to compare in both models.

The result in beam model shows exact values for axial and bending deformations with 0% error value. The angle of twist in beam model shows error value of close to zero with a 0.014%. In 3D solid, the difference is observed highly in the axial deformation showing a 14% error. The 3D solid model in bending deformation shows a 10% error. The error observed in the angle of twist is smaller when compared to other deformation with 1.92 % error. The result for the analysis showed that a beam model shows more accuracy based on the model that is build. The differences in the 3D solid mechanics is seen to have more errors compare to the beam model. The beam model shows a great accuracy and for a simple geometry a better suit for users. The beam model is very simple to design and more precise in the result.

The study and simulation time and physical memory(cost) is also observed. The time simulation shows a difference between both models. The time study is computed in axial, bending and torsion. The beam model is simulated with a lesser time than that of a 3D solid model in the three deformations that is analyzed. The differences are found to be in second. The results didn't show much differences while computation due to the simplified components geometry. A geometry with a complex structure will have a big difference in respect of time computation. The physical memory during the simulation is also observed. The physical memory that a beam model takes is less than that of 3D model.

Finally, the result is obtained and indicated from COMSOL Multiphysics shows a difference in precise, time and physical memory. Considering a simple geometry, it is safe to say that using a beam model when analyzing the materials properties will save a great deal of time and cost for any users while giving accurate values.. The time differences and memory in 3D solid and beam model is close due to the simplicity of the structure.

The time taken to analyze a complex geometry in 3D solids could be weeks or months. The beam model will not only provide a reduced designing process but also saves time. There are different kinds of effects that can be looked at to make comparison between the beam model and 3D solid. It is recommended that a further studies should be conducted with a complex geometries and with a different kinds of effect.

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