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Spherical Trigonometry Handbook for Navigators

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Abstract

The objective of this thesis was to produce a handbook of all the relevant information regarding the mathematics and the practical real-world applications of spherical trigonometry and, more generally, the concept of Great Circle Navigation, relating to the fundamentals behind the practice of navigating ocean-going vessels on the high seas.

While exhaustive research on the subject-matter had already been completed in the centuries past, the need was felt to compile a summarized, compact, version of the overwhelming scientific data, into a handbook that could be used, for instance, by students in Maritime academies, already practicing navigators or anyone interested in navigation as a whole.

As the premise behind the work was to introduce a quick-reference tool, it was self-evident from the start of the project, that certain parts of the theory would have to be left out, such as, the thorough derivation of the formulae used. Only those, whose understanding was felt necessary for the end user, were included. The same basic spherical trigonometry formulae are also used in Celestial Navigation, but as it pertains mostly with position fixing rather than the actual navigation part, it was left out of this thesis.

On the other hand, the concept of Rhumb Line Navigation was felt imperative to be included, as it highlights the difference in distances covered when compared to those by sailing along great circles.

Furthermore, rhumb lines are an integral part of the actual practice of ship navigation in the real world, in that they are combined with great circle navigation in a practice known as Composite Sailing.

Key words
Spherical Trigonometry, Great Circle

CONTENTS

1 INTRODUCTION

Spherical trigonometry plays an integral part in the practice of navigation, for both aircraft and waterborne vessels. It is the fundamental background from which the practical application of the concept of Great Circle Navigation stems from, as the shortest distance between two points on a sphere is located along an arc of a great circle, a fact most crucial to efficient navigation (Bowditch, 3).

Navigation in Maritime Academies is taught to Sea Captain students in several different courses as required study, mandated by the International Maritime Organization (IMO). Although in modern times, voyage planning, of which the concepts discussed in this thesis heavily rely on, is done exclusively using electronic aids and computerized programs, such as the Electronic Chart Display and Information System (ECDIS), it is still considered to be imperative for students to know the basics of manually calculating passages using the aforementioned concepts.

However, it is my opinion, based on personal experience and observation, that the theories behind the applications are not taught exhaustively, but rather the required formulae for the computations are just given to the students to use, without the information-receiving individual necessarily fully understanding the theory, becoming merely a “button-pusher” as opposed to an enlightened navigator grasping the underlying fact, the eternal mathematical truth.

The motivation behind this study was to compile a selection of mathematical formulae and the theoretical background relevant to the actual process of calculating various ocean passage-related problems, be they classroom-, or actual real-world navigation-related in nature, into a compact, user-friendly handbook to be used as a learning aid, and as a quick-reference source.

While the thesis is aimed at students of nautical studies, either in vocational schools, or Universities of Applied Science, it can also be utilized by already practicing navigators to maintain the theoretical know-how behind the practical applications of various navigational techniques, as well as anyone interested in the ancient art of navigation.

It was not the intention of this work to be a product of purely mathematical study, as exhaustive scientific research in the field has been carried out extensively in the centuries, and even, millennia past, but rather to serve as a more in-depth resource as compared to the brief introduction into the subject matter received in a classroom setting.

Understanding the underlying principles of the concepts taught at school, and used behind the modern computerized navigation software, should be a matter of personal and professional pride for anyone engaged in the field of navigation. It is for this purpose, that this thesis has been written.

2 PLANE TRIGONOMETRY

2.1 The Unit Circle and the Pythagorean Theorem

To be able to understand the more complex mathematics and the relationships between different functions, it is perhaps useful to go back to the basics:
The unit circle is a circle defined as having a *radius* of one. This means that the circumference of a unit circle is equal to:

$$C = 2\pi r = 2\pi$$

This means that the relationship of the circumference and the angle of a full circle, 360 degrees, that is to say, dividing the circumference by an arbitrary angle yields a result as the length of the arc of the said angle, in *radians*.
We can use the unit circle as the basis to define the following relationships that we will take a closer look as we progress.

Finding the lengths of the sides of a right triangle, the well-known equation by the Greek mathematician Pythagoras, can be used:

$$a^2 + b^2 = c^2$$

The laws used for solving triangles, other than right triangles, are listed below for reference, but their applications are left out from this thesis, instead focusing on their spherical counterparts.

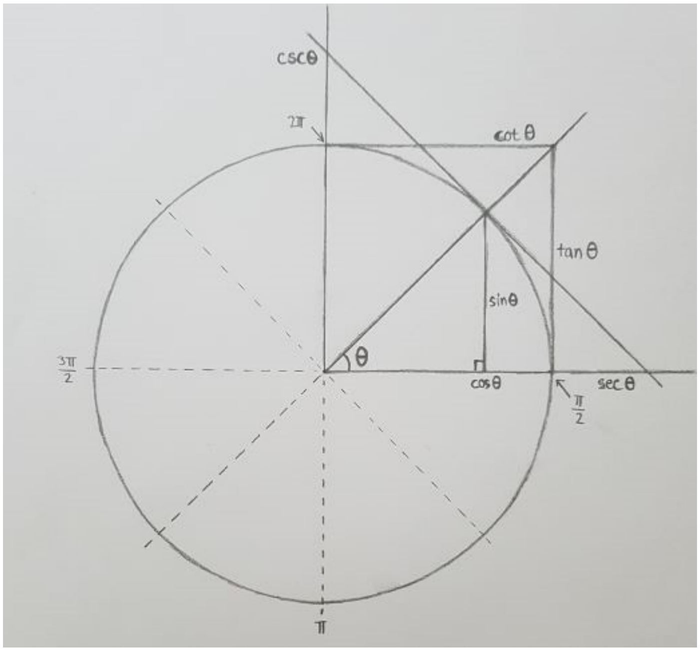

Figure 1. A unit circle with the trigonometric functions (Source: Personal collection).

2.2 Sine, Cosine and Tangent

In a right triangle the sine of an angle yields a ratio of the opposite side with regard to the hypotenuse.
The cosine of the same angle is the ratio of the adjacent side to the hypotenuse.
And finally, the tangent is the result of the division between the opposite and adjacent sides. The tangent can be considered to represent the radius of the earth, as its value is constant irrespective of the angle whereas as the values for the sine and for the cosine change with respect to the angle.

A most useful mnemonic to remember is “SOH CAH TOA”, meaning the following:

$$\text{SOH: Sine} = \frac{Opposite}{Hypotenuse}$$

$$\text{CAH: Cosine} = \frac{Adjacent}{Hypotenuse}$$

$$\text{TOA: Tangent} = \frac{Opposite}{Adjacent}$$

2.2.1 Law of sines

The law is defined as being the relationship of a side of a planar triangle with respect to the sine of its opposite angle.

$$\frac{a}{sin\,\alpha} = \frac{b}{sin\,\beta} = \frac{c}{sin\,\gamma}$$

2.2.2 Law of cosines

$$a^2 = b^2 + c^2 - 2bc\,cos\,\alpha$$
$$b^2 = a^2 + c^2 - 2ac\,cos\,\beta$$
$$c^2 = a^2 + b^2 - 2ab\,cos\,\gamma$$

2.2.3 Law of tangents

$$\frac{a-b}{a+b} = \frac{tan\left[\frac{1}{2}(\alpha-\beta)\right]}{tan\left[\frac{1}{2}(\alpha+\beta)\right]}$$

2.3 Rhumb Line Navigation

Also known as a *loxodrome*, the rhumb line is a line that intersects all of the lines of longitude on its path at the same angle (Bowditch, 3). This is in contrast to the constantly changing angle used when travelling along a great circle.

Suppose you are flying an aircraft with infinite fuel and range eastbound along the Equator. You then set and hold a course of 080°(T), meaning a *true course.* This constant course will eventually lead you into spiralling towards the North Pole. Your track would resemble a loxodrome, hence the term. This somewhat unrealistic example highlights the effects of using a constant course.
Rhumb line navigation is widely used when the distances to be covered are relatively small or in areas where frequent course alterations are required, such as in an archipelago or near-coastal regions.

2.3.1 Plane Sailing and Departure

The concept of plane sailing assumes that the Earth is flat. Obviously we know that this is not the case in reality, but for short distances using rhumb line navigation this assumption can be utilized.

While the assumption of a flat Earth is made in plane sailing, the reality of the parallels of latitude being equidistant from each other along with the fact that the lines of longitude *are not* equidistant, is taken into account.

The meaning of the term *departure* in this context means the difference in longitude. Consider the distance from the Earth's center to the Equator as being the distance R, and another, arbitrarily chosen point on the surface. The arc between these two points subtends an angle at the center of the Earth. This angle represents the latitude, φ, in question. The circumference of the circle at the Equator is $2\pi R$ and the circumference of the small circle at the arbitrary latitude is $2\pi r$, the r representing the cosine of the subtended angle. The ratio of the small circle over the great circle is then just r/R. This yields the value of the departure at the chosen latitude, with respect to the change of longitude at the Equator.

Departure is calculated to convert the differing distances between two lines of longitude into a constant value, to correct for the fact that the distance between two longitudes on the Equator is greater than the distance between the same two longitudes on, for example, on the 60th latitude. In fact, the latter is just half of the former, as the cosine of sixty degrees is one half.

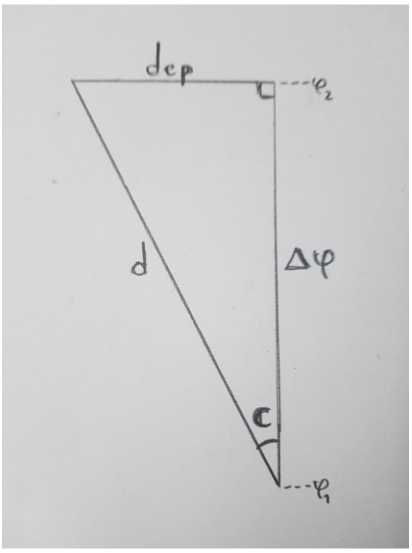

Figure 2. A plane sailing triangle (Source: Personal collection).

$$\frac{2\pi r}{2\pi R} = \frac{r}{R} = \cos\varphi = \frac{dep}{\Delta\lambda}$$

$$\cos\varphi = \frac{dep}{\Delta\lambda}$$

$$\Delta\lambda = \frac{dep}{\cos\varphi}$$

$$dep = \cos\varphi\Delta\lambda$$

When calculating rhumb line distances, a method known as a mean latitude can be used. Like its name suggests, it is the mean value of two latitudes, such as the latitude of your present location and the latitude of your intended location. Using the mean latitude method, accurate-enough results can be expected to around 600 nautical miles.

2.3.2 Mercator Sailing and Meridional Parts

When rhumb line distances exceed 600 nautical miles, an alternate method of calculation should be utilized. In this method the latitude difference is converted into so-called *meridional parts,* the values of which are given in minutes corresponding to one minute of arc along the Equator.

The resulting triangle can be drawn onto a Mercator chart, hence the name Mercator sailing. As is the case on a Mercator chart, the lines of longitude are all equidistant, however, the parallels of latitude are not, as their distance increases towards the poles to take into account for the fact that the spheroidal Earth is projected on a flat surface. The formula for calculating the meridional part of a longitude is the following:

$$(360°/2\pi \ln \tan (45+\varphi/2))60 - 23 \sin \varphi$$

The meridional parts for any given latitude can also be found in proper nautical tables, but it is useful to memorize the above formula.

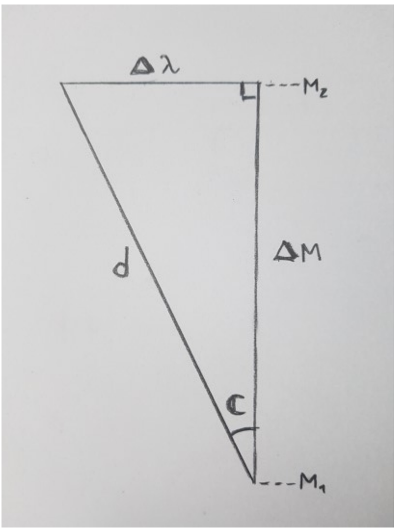

Figure 3. A Mercator sailing triangle (Source: Personal collection).

2.3.3 Rhumb Line Course and Distance

To calculate the rhumb line course and distance, the following pieces of information must first be obtained:

$\Delta\varphi$	the difference of the departure and arrival latitudes
$\Delta\lambda$	the difference of the departure and arrival longitudes
ΔM	the difference of the meridional parts of the respective latitudes

$$\tan C = \frac{\Delta\lambda}{\Delta M}$$

in the above, the inverse tangent of the angle C is the internal angle at the point of origin and its value is used in the following formula $d = \frac{\Delta\varphi}{\cos C}$ (with five decimal points of accuracy) to calculate the rhumb line distance.

In order to find out which course to steer, C_n, one needs to identify the direction of travel: if heading between North and East, the course to steer is the same as C, if heading between East and south, the course to steer is
180°- C, if heading between South and West, C is added to 180 degrees, and finally when heading between West and North, C is deducted from 360 degrees.

3 SPHERICAL TRIGONOMETRY

3.1 The Unit Sphere

The unit sphere, like the unit circle, has a radius of one, however a triangle drawn upon it is defined as having six individual angles, as opposed to the three angles required to define a triangle in the two-dimensional circle.

These triangles and their angles are a focal point in this thesis, and as such, they will be scrutinized later on.

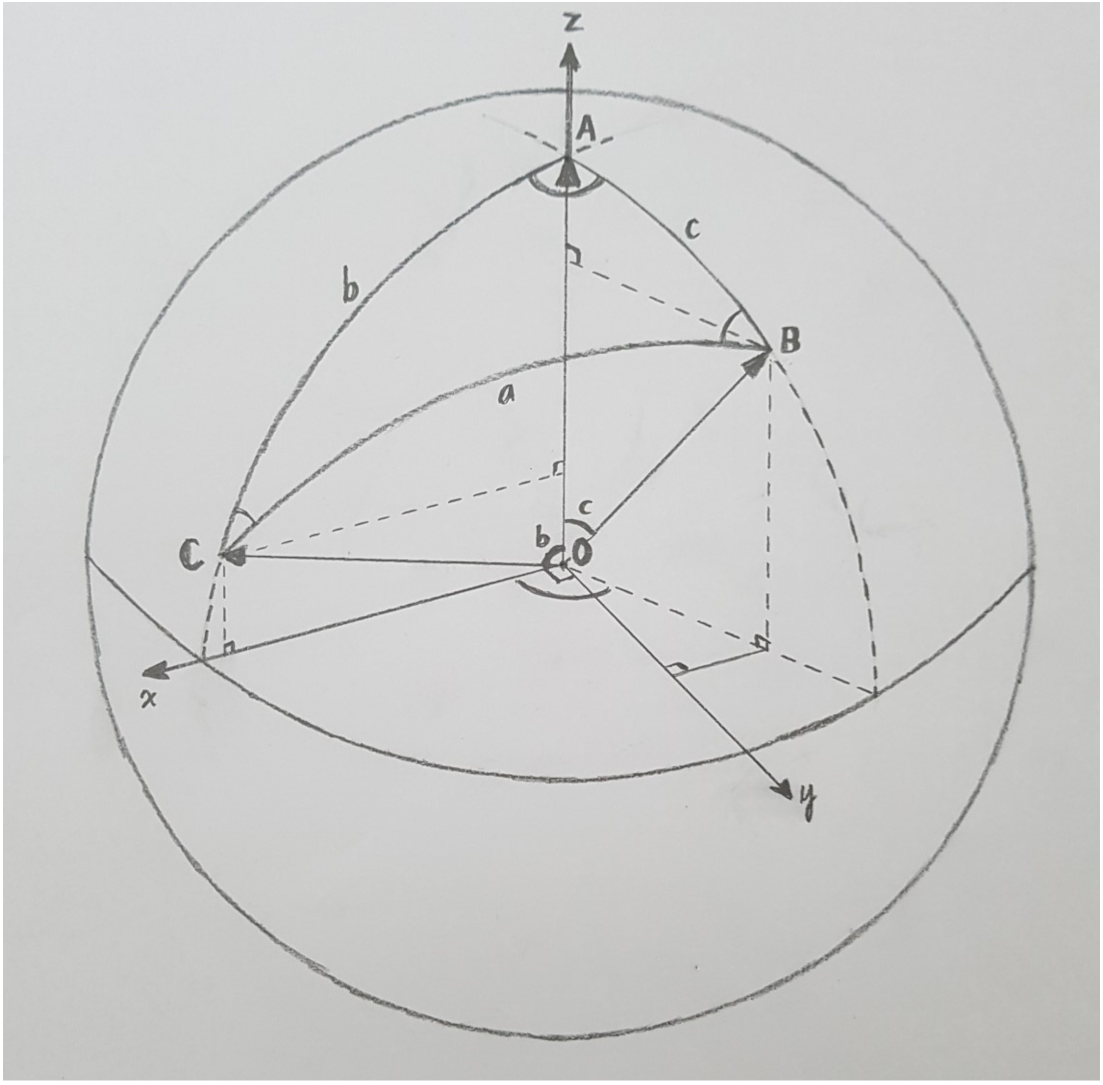

Figure 4. A sphere with a spherical triangle (Source: Personal collection).

3.2 The Great Circle

A great circle is defined as being any such circle that splits a sphere into two equally large parts. On the Earth such great circles are all of the meridians, of which there are an infinite number of, meaning the circles connecting the north and south poles. Meridians are used to define the longitude of a given position on a sphere.
The only great circle of latitude is the Equator, all other parallels of latitude are what are known as small circles (Todhunter, 2).

An important concept in navigation is the relationship between a minute of arc along a great circle on the surface of the Earth, *and* a nautical mile: the historical definition of the nautical mile (1,852 meters in SI-units) is that it is equal to a minute of latitude along a line of longitude. One degree is equal to sixty minutes, so the distance of one degree of latitude is equal to sixty nautical miles, subsequently the circumference of the Earth is (roughly) 21,600 nautical miles. If you cover a distance of one nautical mile in sixty minutes, you have travelled at the speed of one *knot*, the nautical unit of speed. All of the above units (minute, nautical mile and the knot) can be, and frequently are, denoted by the sign " ′ ", and hereinafter that notation is used to describe any of the aforementioned identities.

As mentioned before, the shortest distance between two points on a sphere is along an arc of a great circle. It would be beneficial for any vessel to exclusively navigate along such great circles, however, that is not always practical due to several reasons discussed further in the practical applications part of the thesis.

3.3 Spherical Triangles

A spherical triangle is formed at the intersection of three great circles. Such triangles and their properties are what the calculations regarding great circle navigation are all based on.
Consider a spherical triangle originating from the north pole on a sphere, *a polar spherical triangle*. Its two sides extending from the pole are arcs of a great circle, meridians, and they represent lines of longitude, λ, and the angle between the two sides

defines the change in longitude, $\Delta\lambda$. The third and final side of the triangle represents the distance between points located on the meridians. These points are at the intersections of longitude and latitude, φ, and they represent the departure and arrival co-ordinates, respectfully.

As mentioned before, a spherical triangle can be defined as being the product of six different angles, three of which are projected from the center of the sphere onto the surface, and the three angles between the sides of the triangle.

Planning a voyage along a great circle requires the navigator to solve a spherical triangle, or several such, if the navigator wants to know, for example, specific latitude- and longitude-crossings along the route. We are fortunate in that the great mathematical minds of the past have derived all of the relevant rules regarding the solving of such triangles. Nevertheless, I find it important that we take a closer look into the fascinating world of spherical trigonometry, to be able to fully understand and appreciate their efforts that have greatly benefitted mankind.

We will now take a look at the different types of spherical triangles. The specific solutions to the different triangles will be examined in the solutions part of the research.

3.3.1 Oblique Spherical Triangle

A spherical triangle is said to be oblique if none of its included angles is 90 degrees, or two or three of its included angles are 90 degrees.

3.3.2 Right Spherical Triangle

If a spherical triangle has one included 90-degree angle, it is said to be a right spherical triangle.

3.3.3 Quadrantal Spherical Triangle

Not to be confused with the above, a quadrantal spherical triangle is one with one *side* being equal to $\frac{\pi}{2}$, that is to say, 90 degrees.

3.4 Laws and Rules for Solving Spherical Triangles

As with a planar triangle, their spherical counterparts have specific laws and rules to be used when solving their properties.
The following two such laws are focused on hereinafter.

3.4.1 The Spherical Law of Sines

Similar to the planar law of sines, its spherical counterpart also gives a ratio, but in this case the ratio is of the side over the angle.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

3.4.2 The Spherical Law of Cosines

The most widely used formulae when dealing with spherical triangles are the following cosine laws.

$$\cos c = \cos a \cos b + \sin b \; sinc \cos C$$

$$\cos C = \; - \cos A \cos B + \sin A \sin B \cos C$$

The special case arises when $C = \frac{\pi}{2}$, from this it follows that $cos\ c = 0$

In this case when can simply use the formula

$$\cos c = \cos a \cos b$$

The following alternate formulae can also be used

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}$$

3.4.3 Napier's Rules

Quadrantal and right spherical triangles can be solved using the rules coined by the Scottish mathematician John Napier (1550-1617). In his rules he was able to combine the previously used ten separate trigonometric equations into only two. To solve the aforementioned spherical triangles using his rules a diagram is also drawn, known as a *Napier's Circle*. Such a circle will be constructed further in the text below.

Consider a right spherical triangle *CAB*, the opposite sides being *c*, *a*, and *b*, respectively. Angles *C* and *B* are known, *B* being the right angle. Side *b* is also known. Constructing a Napier's Circle starts by drawing a circle, on top of which, on the outside, the notation of the right angle is drawn. The diagram is split into two hemispheres, the top one is then divided into two equal parts, and the lower hemisphere is divided into three parts.

On the upper hemisphere, the other quadrant is marked as *a*, and the other as *c*. They are the adjacent parts to the right angle, meaning the sides *a* and *c*.

On the lower hemisphere, the part opposite the right angle is marked as its complementary angle, that is to say, 90 degrees minus the opposite part, in this case the side *b*. The notation is therefore 90°-*b*, or $\bar{b}$. The other two opposite parts are labelled similarly in their own corresponding slots using their complementary angles, meaning 90°-*C*, or $\bar{C}$, and 90°-*A*, or $\bar{A}$, respectively. The circle is now complete.

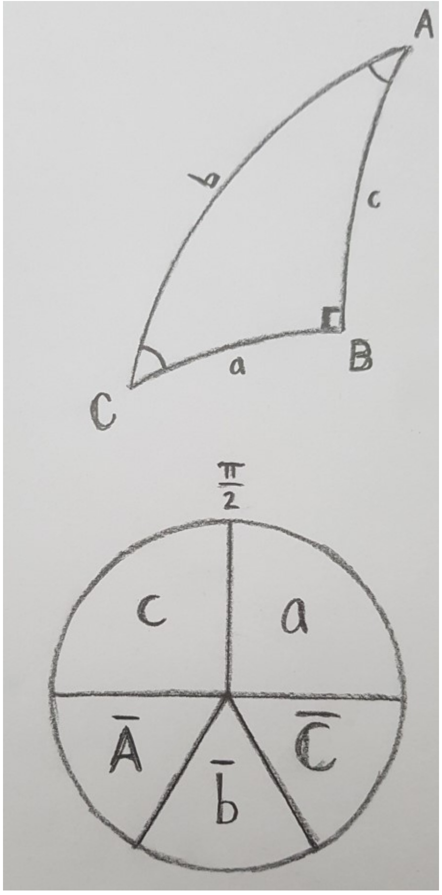

Figure 5. A right spherical triangle and a Napier's Circle (Source: Personal collection).

Napier has the following two rules:

1. The sine of a part is equal to the product of the tangents of the two adjacent parts.

2. The sine of a part is equal to the product of the cosines of the two opposite parts.

The first one means that the sine of any *middle* part is equal to the cosines of the opposite parts, the latter meaning that the sine of any middle part is equal to the tangents of the adjacent parts. To be able to use either of the rules, one much choose which part is to be used as the middle part, thereafter it is just a matter of applying the applicable rule.

A similar diagram can be constructed to be used in solving a quadrantal spherical triangle. The only relevant difference is to substitute the 90-degree angle for a 90-degree side, the adjacent angles are then input as the adjacents in the diagram, and then finally the complementary lengths (in degrees) of the opposite two sides and the complementary angle of the remaining angle are then added to the bottom.

The same rules as with the right spherical angle apply to the case of the quadrantal spherical angle. Again, the choice of the middle part is needed to use the rules.

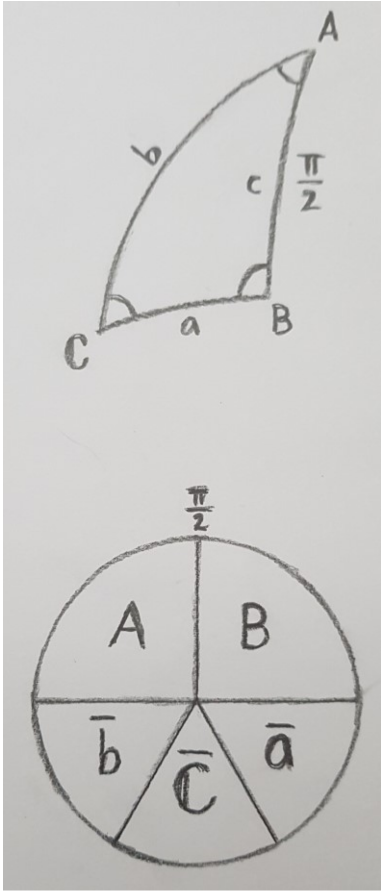

Figure 6. A quadrantal spherical triangle and a Napier's Circle (Source: Personal collection).

3.5 Great Circle Navigation

As mentioned, sailing along a great circle yields the shortest distance on a sphere, and for all intents and purposes, the Earth can be assumed to be a perfect sphere when dealing with navigational calculations, as the error incurred is insufficient to cause any actual problems.

Solving navigational problems with regard to great circle navigation begins by collecting all of the relevant data at your disposal. Consider the following: you know your present location and its latitude and longitude, φ_1 and λ_1, respectfully. You also know the co-ordinates of your desired destination (φ_2 and λ_2). Using these initial values you will then calculate the difference in longitude, $\Delta\lambda$. Once you have the longitude difference calculated, then it becomes a simple task of applying a version of the spherical law of cosines (to be shown later in the text) to find the great circle distance (in minutes). To convert the value into nautical miles, multiplying by sixty is required. The second value to solve for is the initial great circle angle. By using another variant of the same cosine laws, (again, an example calculation follows further), this value will be obtained. Let's call this angle C. Converting this internal angle into a course, one needs to take into consideration the direction of the intended travel: if you are eastbound, then the course to set, C_n, is the same as C, if due West, deduct the angle from 360 degrees.

The great circle route can take a vessel into potentially hazardous latitudes. Such a problem arises when the departure and arrival latitudes are already reasonably high, yet close to each other, and the longitude difference is great.

3.5.1 Point of Vertex

The point along the arc of a great circle that is the closest to either pole is known as the point of vertex. Determining its position is useful for a navigator, in that by doing so, the navigator can verify if the intended route takes the vessel to an unreasonably high latitude, where the prevailing weather conditions might be unfavorable for efficient navigation. If the point of vertex lies on the route and the latitude is deemed to be too high, alternate solutions for the route might have to be considered.
The actual location of the vertex point can be found using Napier's rules, as a great circle reaching its maximum latitude forms a right angle with a specific meridian, forming a solvable right spherical triangle. However, perhaps for convenience, formulae have been derived for finding the constituent parts for defining the location of the point of vertex, and they are as follows:

$$\cos\varphi_V = \sin C \cos\varphi_1$$

$$tan\,\Delta\lambda_v = \frac{1}{tan\,C\,sin\,\varphi_1}$$

$$tan\,d_v = \frac{cos\,C}{tan\,\varphi_1}$$

where *C* is the initial great circle course and φ_1 is the departure latitude.

3.5.2 Composite Sailing

If for some reason the navigator does not want to sail exactly along an arc of a great circle, the practice of composite sailing can be used instead. It involves adding several rhumb line legs with the individual waypoints being located along the original great circle. A variation of composite sailing adds a particular latitude that the navigator does not want to cross. If this approach is used, then the route looks like something like the following:
First the “no-go latitude” is chosen. A great circle that goes between the departure location and having its tangent (i.e. having its point of vertex) on the chosen parallel of latitude produces the arc that is followed using a pre-determined number of rhumb line legs.
Napier’s rules can once again be used to find the distance to the point of vertex, but the following formula derived from said rules can be used to find the longitude difference to the point of vertex:

$cos\,\Delta\lambda_v = \frac{tan\,\varphi_1}{tan\,\varphi_m}$ where φ_1 is the departure latitude and φ_m is the maximum latitude, i.e. the latitude that we do not want to cross.

To find the other point of vertex, simply substitute the departure latitude by the arrival latitude.
Usually one waypoint at the mid-point of the arc is enough, so the first two legs of the composite sailing passage is from your departure location to the first waypoint, on a loxodrome, at which point the loxodrome course is altered, towards the point of vertex. Once the desired latitude is reached, the course is altered to sail along the parallel of

latitude (due East or West depending on where you are going), until the second point of vertex (the tangent of a great circle that goes between your destination) is reached. Course is then altered towards the final course alteration point, at the midpoint of the great circle arc, mirroring the procedure used in the beginning. The aforementioned ocean passage would then constitute of five separate loxodrome legs with four course changes along the way. The total distance travelled would obviously be somewhat greater than had a direct great circle route been used instead, however, the unfavorably high latitudes were avoided, and the distance covered would be considerably less than had a pure loxodrome route been used.

3.5.2.1 Intermediate Latitudes and Longitudes

To find the latitudes for course-alterations when using composite sailing, two approaches can be used. Unsurpisingly, the first one deals with constructing an oblique spherical triangle from the right spherical triangle used to define the point of vertex. The choice of the longitude for the course changes is then made. This longitude, known as the intermediate longitude, λ_i, forms the polar angle of the triangle. The latitude for the course change is then found by using the spherical law of cosines.
As an alternate method, a formula derived from the solutions for spherical triangles, and it is as follows:

$$tan\, \varphi_i = \frac{tan\, \varphi_2\, sin(\lambda_i - \lambda_1) - tan\, \varphi_1\, sin(\lambda_i - \lambda_2)}{sin(\lambda_2 - \lambda_1)}$$

where φ_1 is the departure latitude
φ_2 is the latitude of the first point of vertex
λ_1 is the departure longitude
λ_2 is the longitude for the first point of vertex
φ_i is the intermediate longitude at which the course is to be altered

4 EXAMPLE CALCULATIONS

4.1 Example 1

A ship is due to sail from Miyazaki, Japan $\varphi\ 31°\ 55{,}6'N\ \lambda\ 131°\ 29{,}2'E$ to Valparaiso, Chile $\varphi\ 33°\ 01{,}0'S\ \lambda\ 071°\ 38{,}3'W$, using a great circle route. Calculate the following:

a) Great Circle Distance
b) Initial Great Circle Course
c) Equator-crossing longitude
d) The Antimeridian-crossing latitude

The solutions are as follows:
As is the case with most of these types of problems, it is a good idea to draw a picture of the initial condition and gather all of the information given. From the premise we can see that our departure location is in the Northern Hemisphere and our destination is in the Southern Hemisphere. This, coupled with the fact that our destination also has a longitude *west* from the Prime Meridian. Both of these facts, the southern latitude and the western longitude require us to input their values as *negatives* in our formulae, as we will construct a spherical triangle, and they are, by convention, drawn from the North Pole. By giving the aforementioned values a negative sign, we will get the output from our calculations as positive values.
To calculate the great circle distance we will use the Spherical Law of Cosines, using the following notation

$$\cos d = \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 \cos \Delta\lambda$$

where φ_1 and φ_2 are the departure and arrival latitudes, respectively, and $\Delta\lambda$ is their longitude difference ((180°-131°29,2’ E) - (180°-71°38,3’ W) = 156°52,5’)

cos d = sin 31°55.6’ sin -33°1’ + cos 31°55.6’ cos -33°1’ cos 156°52.5’

d = 160.49919° (multiply by sixty to get the distance in nautical miles)

d = $9630'$

To calculate the initial great circle course we will use one of the cosine laws for spherical triangles as discussed earlier;

$$cos\, C = \frac{sin\, \varphi_2 - sin\, \varphi_1\, cos\, d}{cos\, \varphi_1\;\, sin\, d}$$

$$cos\, C = \frac{sin\, -33°1' - sin\, 31°55.6'\, cos\, 160.49919°}{cos\, 31°55.6\;\, sin\, 160.49919°}$$

with the great circle distance being input into the formula in minutes, five decimal places is sufficient for an accurate enough result.

By taking the inverse cosine of the value, we get the initial angle "*C*" to be 99.422827°, or, 99° 25.4' using the degrees-and-minutes notation.

Next we are asked to find the Equator-crossing longitude. We do it by constructing and solving a spherical triangle, that being a part of the original spherical triangle.

Again, as with any polar spherical triangle, it is constructed from the North Pole, we will again label it "*A*". Side "*b*" is again the side between "*A*" and "*C*", and the value of the angle *C* was just calculated. Side "*a*" represents the distance to the Equator (which we are not necessarily interested in), being opposite of the angle "*A*", the value of which we *are* interested in, as when we have its value, we can find the longitude of the Equator crossing. The final side, side "*c*", is opposite to angle "*C*", and since the side originates from the North Pole ending at the Equator, its length is just simply $\frac{\pi}{2}$, or 90 degrees. This means that the triangle in question is a quadrantal spherical triangle, and to solve it, Napier's rules can be utilized.

As we already know the value for angle *C* (the initial great circle course), and the length of the side b $\left(\frac{\pi}{2} - \varphi_1\right)$, we can use these two values as our known values in the Napier's diagram.

Let's use the sine of complementary angle of side b as our middle part, and the tangents of the complementary angles of angles *C* and *A*. One specific rule is to put a minus sign onto the right side of the equation, *if both of the parts are either angles or sides*.

$$sin\,\bar{b} = -\,tan\,\bar{C}\,tan\,A$$
$$sin\,31°55.3' = -\,tan\,9°\,25.4'\,tan\,A$$
$$tan^{-1} = -\frac{sin\,31^{\circ}55.6'}{tan\,9°\,25.4'}$$
$$A = 72°34.6'$$

Angle *A* represents the longitude difference ($\Delta\lambda$) between our departure location and the intermediate part of our journey, the Equator-crossing longitude asked in the problem. Adding the value of $\Delta\lambda$ (72° 34.6') to our departure longitude (131° 29.2'E) we get a value of over 180 degrees, so we know that our Equator crossing takes place on the Western hemisphere. To get the correct longitude value, we must deduct this value from 360 degrees to get the final longitude of 155° 56.2'W.

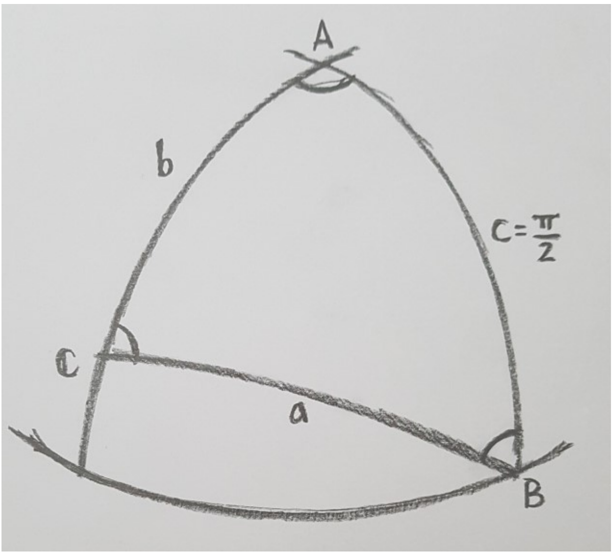

Figure 6. A quadrantal spherical triangle (Source: Personal collection).

To calculate the latitude at which the antemeridian (λ 180°) is crossed, we can construct another spherical triangle, again using parts of our original triangle, namely angles *C* and side *b*. We can find the angle *A´* by deducting 180 degrees from our departure longitude (180° - 131° 29.2' = 48° 30.8'). We then have a spherical triangle of which we know the values for two angles and the length of the side between them. Figure 7 illustrates this condition.

By solving the length of the side c we can determine the latitude in question. To do this we must first calculate the value for angle B'. We do this by using a modified formula of the cosine law for spherical triangles.

$$B' = cos^{-1}(sin\, A'\, sin\, C\, cos\, b - cos\, A'\, cos\, C)$$

By inputting our newly-calculated values into the above equation, we find the value for angle B' to be 60° 2.9’.

We can now use the spherical law of sines to determine the value for side c (and, if we’d like, for side *a* as well).

$$c = sin^{-1}\left(sin\, C\, \frac{sin\, b}{sin\, B'}\right)$$

The length of the side *c* is 75° 5.4’. By deducting this value from 90° we get the latitude for the crossing of the antimeridian, 14° 54.6’ N.

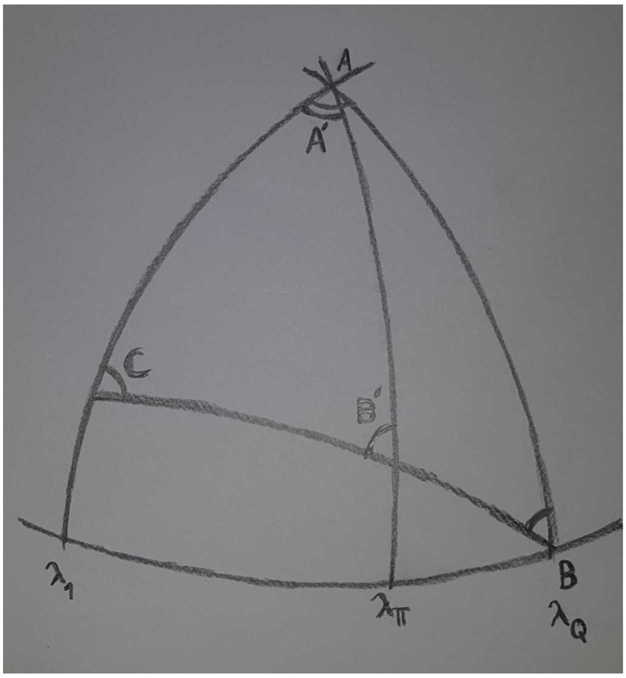

Figure 7. A spherical triangle constructed from a quadrantal spherical triangle (Source: Personal collection).

4.2 Example 2

A ship is departing Christchurch, New Zealand φ 43° 31.8' S λ 172° 37.2' E
for Valparaiso, Chile φ 33° 01.0' S λ 071° 38.3' W.
Calculate the following:
- great circle distance
- point of vertex and the distance to it
- rhumb line distance
- composite sailing distance, when φ 50° 00.0' S is not crossed, and course is changed at mid-longitudes
- compare the resulting distances

The great circle distance:

cos d = sin -43° 31.8' sin -33° 1' + cos -43° 31.8' cos -33° 1' cos 115° 44.5'

d = 83.61287° x 60 = 5017'

The initial great circle course:

$$\cos C = \frac{\sin -33°1' - \sin -43°31.8' \cos 83.61287°}{\cos^{-43^0} 31.8' \cos 83.61287°}$$

C= 130.53432°

Finding the point of vertex:

cos φ_v = sin 130.53432° cos -43° 31.8'

φ_v = 56° 33.8' S

tan $\Delta\lambda_v = \frac{1}{\tan 130.53432° \sin -43°31.8'}$

$\Delta\lambda_v$ = 51°9.1'

$\lambda_v = \lambda_1 + \Delta\lambda_v$

λ_v = 172° 37.2' + 51°9.1'

λ_v = 223° 46.3' as the value exceeds 180°, deduct it from 360°

λ_v = 136° 13.7’ W

Point of vertex:

φ 56° 33.8’ S λ 136° 13.7’ W

Distance to the point of vertex:

$tan\, d_v = \frac{cos\, 130.53432°}{tan\, -43°31.8'}$ multiply by 60 to get the distance in nautical miles

d_v = 2063’ the point of vertex is ahead, 2063’ away

Rhumb Line Distance:

$\Delta\varphi = 630.8'$

$\Delta\lambda$ = 6944.5’

M_1 = 2890.9’

M_2 = 2088.2’

ΔM = 802.7’

$tan\, C = \frac{\Delta\lambda}{\Delta M}$

$tan^{-1} C = \frac{69445'}{802.7'}$

$C = 83.40656°$ since we are sailing due East, our C = C_n

$d = \frac{\Delta\varphi}{cos\, C}$

$d = \frac{630.8'}{cos\, 83.40656°}$

$d = 5494'$

Composite sailing:

The longitude difference to the first point of vertex (at the maximum-desired latitude) must be calculated. The other $\Delta\lambda_v$ can be calculated at this point also, if desired.

$cos\, \Delta\lambda_{v_1} = \frac{tan\, -43°\, 31.8'}{tan\, -50°}$

$\Delta\lambda_{v1}$ = 37° 8.7’ this value is added to the departure longitude

λ_{v1} = 209° 45.9’ as the value exceeds 180°, this value must be deducted from 360° to get the correct longitude

λ_{v1} = 150° 14.1’ W

$cos\, \Delta\lambda_{v_2} = \frac{tan\, -33°\, 1'}{tan\, -50°}$

$\Delta\lambda_{v2}$ = 56° 57.4’ this value is added to the arrival longitude

λ_{v2} = 128° 35.7’ W

Determining the intermediate latitudes and longitudes:

The course is to be changed at mid-longitudes, that is to say, between the departure and arrival longitudes and the longitudes of both of the points of vertex. Finding these longitudes is a simple matter of taking the mean value of the longitudes in question:

λ_{i_1} = 168° 48.5’ W

λ_{i_2} = 100° 7’ W

As mentioned earlier, finding the intermediate latitudes can be accomplished either by using Napier’s, or the specific formula derived from said rules. For this example, the latter is to be used.

$$tan\, \varphi_i = \frac{tan\, \varphi_2\, sin(\lambda_i - \lambda_1) - tan\, \varphi_1\, sin(\lambda_i - \lambda_2)}{sin(\lambda_2 - \lambda_1)}$$ where

φ_1 = departure latitude

φ_2 = latitude of the first point of vertex

λ_1 = departure longitude

λ_2 = longitude of the first point of vertex

λ_i = first intermediate longitude

φ_i = first intermediate latitude

$$tan\, \varphi_{i1} = \frac{tan\, -50°\, sin(-168°\, 48.5' - 172°37.2') - tan\, -43°\, 31.8'\, sin(-168°\, 48.5' - (-150°\, 14.1'))}{sin(-150°\, 14.1' - 172°\, 37.2')}$$

φ_{i_1} = 48° 29.1’ S

The second intermediate latitude is calculated similarly, however the following substitutions are made:

φ_1 = latitude of the second point of vertex

φ_2 = arrival latitude

λ_1 = longitude of the second point of vertex

λ_2 = arrival longitude

λ_i = second intermediate longitude

φ_i = second intermediate latitude

$$tan\,\varphi_{i2} = \frac{tan\,-33°1'\,sin(-100°\,7'-(-128°35.7')) - tan\,-50°\;sin(-100°\,7'-(-71°\,38.3'))}{sin(-71°\,38.3'-(-128°\,35.7')}$$

φ_{i2} = 46° 19.8' S

We have now calculated the both of the intermediate positions, representing the first and last course-alteration positions:

i_1 = φ 48° 29.1' S λ 168° 48.5' W

i_2 = φ 46° 19.8' S λ 100° 07.0' W

Now all that remains is to calculate the individual rhumb line legs to get the composite sailing distance.

Leg 1:

From φ 43° 31.8' S λ 172° 37.2' E

To φ 48° 29.1' S λ 168° 48.5' W

$\Delta\varphi$ = 4°57.3', or 297.3' $\Delta\lambda$ = 18° 34.3', or 1114.3'

Meridional parts are then either calculated using the formula

(360°/2π ln tan (45+φ/2))60 – 23 sin φ

or by finding their values from literature

M_1 = 2890.9'

M_2 = 3318.0'

ΔM = 427.1'

$$tan\,C = \frac{\Delta\lambda}{\Delta M}$$

$$tan^{-1}C = \frac{1114.3'}{427.1'}$$

$$C = 69.02867°$$

$d = \frac{\Delta\varphi}{\cos C}$

$d = \frac{297.3'}{\cos 69.02867°}$

$d_1 = 831'$

Leg 2:

From φ 48° 29.1' S λ 168° 48.5' W

To φ 50° 00.0' S λ 150° 14.1' W

$\Delta\varphi$ = 1°30.9', or 90.9' $\Delta\lambda$ = 18° 34.3', or 1114.4'

M_1 = 3318.0'

M_2 = 3456.8'

ΔM = 138.8'

$\tan C = \frac{\Delta\lambda}{\Delta M}$

$\tan^{-1} C = \frac{1114.4'}{138.8'}$

$C = 82.90030°$

$d = \frac{\Delta\varphi}{\cos C}$

$d = \frac{90.9'}{\cos 82.90030°}$

$d_2 = 736'$

Leg 3:

From φ 50° 00.0' S λ 150° 14.1' W

To φ 50° 00.0' S λ 128° 35.7' W

$\Delta\lambda$ = 21°38.4', or 1298.4'

$dep = \cos\varphi\Delta\lambda$

$dep = \cos -50°\ times\ 1298.4'$

$dep = d_3 = 835'$

Leg 4:

From φ 50° 00.0' S λ 128° 35.7' W

To φ 46° 19.8’ S λ 100° 07.0’ W

$\Delta\varphi$ = 3°40.2’, or 220.2’ $\Delta\lambda$ = 28° 28.7’, or 1708.7’

M_1 = 3456.8’

M_2 = 3127.4’

ΔM = 329.3’

$tan\, C = \frac{\Delta\lambda}{\Delta M}$

$tan^{-1} C = \frac{1708.7'}{329.3'}$

$C = 79.09172°$

$d = \frac{\Delta\varphi}{cos\, C}$

$d = \frac{220.2'}{cos\, 79.09172\,°}$

$d_4 = 1164'$

Leg 5:

From φ 46° 19.8’ S λ 100° 07.0’ W

To φ 33° 01.0’ S λ 071° 38.3’ W

$\Delta\varphi$ = 13°18.8’, or 798.8’ $\Delta\lambda$ = 28° 28.7’, or 1708.7’

M_1 = 3127.5’

M_2 = 2088.2’

ΔM = 1039.3’

$tan\, C = \frac{\Delta\lambda}{\Delta M}$

$tan^{-1} C = \frac{1708.7'}{1039.3'}$

$C = 58.69035°$

$d = \frac{\Delta\varphi}{cos\, C}$

$d = \frac{798.8'}{cos\, 58.69035°}$

$d_5 = 1537'$

$\Sigma_d = d_1 + d_2 + d_3 + d_4 + d_5$

Σ_d = <u>5103'</u>

Great Circle Distance:	5017'
Composite Sailing Distance:	5103'
Rhumb Line Distance:	5494'

Conclusion:

The composite sailing distance is almost 400 nautical miles shorter than the rhumb line distance, and only slightly longer than the direct great circle route and it avoids the high latitudes where the latter would take the ship to!

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