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Multiple Input Multiple Output (MIMO) Operation Principles

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The main purpose of this project was to explain and illustrate the operation principles of MIMO channel technology and how it works, using examples.

In this project, examples of mathematical calculations and self-developed algorithms were used in explaining the various principles of how a MIMO channel could be modelled. Graphs were also presented to give a pictorial image of the various MIMO channel characteristics and principles.

The project intended to make the operation of MIMO clear with the help of mathematics and simulations. This is because the operation of MIMO is almost impossible to understand without comprehensive mathematics and simulations. Hence mathematical examples used in this project were simple and easy to follow, thereby giving a clear understanding of how a MIMO channel works.

The results showed that theoretically the MIMO channel technology has the ability to increase the capacity of a wireless communication link. It also showed the effects of antennas which are important in implementing the MIMO technology in communication systems to deliver the required capacity. The results also showed that, in order to model a MIMO channel, the transmitted training signals have to be linearly independent. After the channel modelling, unknown signals (dependent or independent signals) can be transmitted over the modelled channel.

The project is useful since it gives any reader the basic idea of how a MIMO communication systems works. Many communication systems such as wireless local area network routers are now experiencing the implementation of MIMO technology and this project provides the reader the benefit and idea about MIMO systems.

Keywords	MIMO channel, orthonormal basis, transmitted signals, received signals

Contents

Abbreviations

$h_{i,j}$	Element (*i*,*j*) of channel matrix $\mathbf{H}$
$\mathbf{x}_i$	Signal transmitted from antenna *i*
$\mathbf{y}_j$	Signal received from antenna *j*
P_s	Signal power (*S/N*)
AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
dB	Decibel
$\mathbf{det}(\mathbf{H})$	Determinant of matrix $\mathbf{H}$
GSM	Global System for Mobile communications
$\mathbf{H}$	Channel matrix (*N* x *M*)
$\mathbf{H}^{-1}$	Inverse of matrix $\mathbf{H}$
$h_n(t)$	Transmitting channel characteristic in time domain
$\mathbf{H}^T$	Transpose of matrix $\mathbf{H}$
$\mathbf{I}_M$	*M* x *M* identity matrix
LOS	Line Of Sight
M	Number of receiving antennas
MIMO	Multiple Input Multiple Output
N	Number of transmitting antennas
NLOS	Non Line Of Sight
OFDM	Orthogonal Frequency Division Multiplexing
QAM	Quadrature Amplitude Modulation
QoS	Quality of Service
$R_j(t)$	*j*-th element of receive antenna as a function of time
RX	Receiver
S	Source symbols
S/N	Signal to Noise ratio
SISO	Single Input Single Output
$T_i(t)$	i-th element of transmit antenna as a function of time
TX	Transmitter
$\mathbf{U}_x$	Orthonormal basis (*K* x *K*) of transmission channel
$\mathbf{W}_c$	Transfer channel matrix (*N* x *M*)
WLAN	Wireless Local Area Network
$\mathbf{W}_x$	Transmitted signals, columns of $\mathbf{X}$, expressed by projections onto orthonormal axes of $\mathbf{U}_x$ (*K* x *N*)

$\mathbf{W}_x^+$	Pseudo inverse of $\mathbf{W}_x$
$\mathbf{W}_y$	Received signals, columns of $\mathbf{Y}$, expressed by projections onto orthonormal axes of $\mathbf{U}_x$ (K x M)
$\mathbf{X}$	Transmitted signal or baseband signals from multiple antennas
$\mathbf{Y}$	Received signal from multiple antennas

1 Introduction

The demand of high bit rate has increased in recent wireless communication networks. Theories by various engineers have proven that the Multiple Input Multiple Output (MIMO) technology has the ability to improve the problem of traffic capacity in the wireless networks. MIMO systems can be defined as the use of multiple antennas at both the transmitting and receiving ends of a wireless communication network. The systems take advantage of multipath transmission paths.

Although various efforts have been made by engineers to improve the data rate, the capacity is never enough for users. Users of mobile wireless devices like to be able to use their devices in streaming live programs, playing more online games and streaming an online movie which involves a high data rates. Telecommunication companies and Internet Service Providers (ISPs) as example in Africa find it difficult to provide high data rate Internet services to their network users, especially mobile users, due to environmental factors. The only option to most of these companies is to provide Internet with a high data rate wirelessly. With the limited bandwidth in space, MIMO technology will be of great benefit to these companies in providing high data rate Internet services to their customers.

Currently cellular systems, such as the third generation (3G) cellular system, satellite communication systems and video broadcasting systems have experienced a great increase in capacity in the implementation of MIMO channel technology. Access point devices such as wireless local area networks (WLAN) routers have also experienced a great change in transmission techniques, with a few using MIMO technology. The main goal of this project is to explain and illustrate the operation of MIMO channel technology.

2 Background of Multiple Input Multiple Output

2.1 Introduction

The multiple input multiple output channel technology is aimed to increase the capacity in the wireless communication network. With the invention of MIMO, the technology seems to gain popularity as it is being implemented in the current commercial wireless products and networks such as broadband wireless access systems, wireless local area networks (WLAN), 3G networks, etc. [1] Figure 1 shows a line of sight (LOS) antenna setup of a MIMO system.

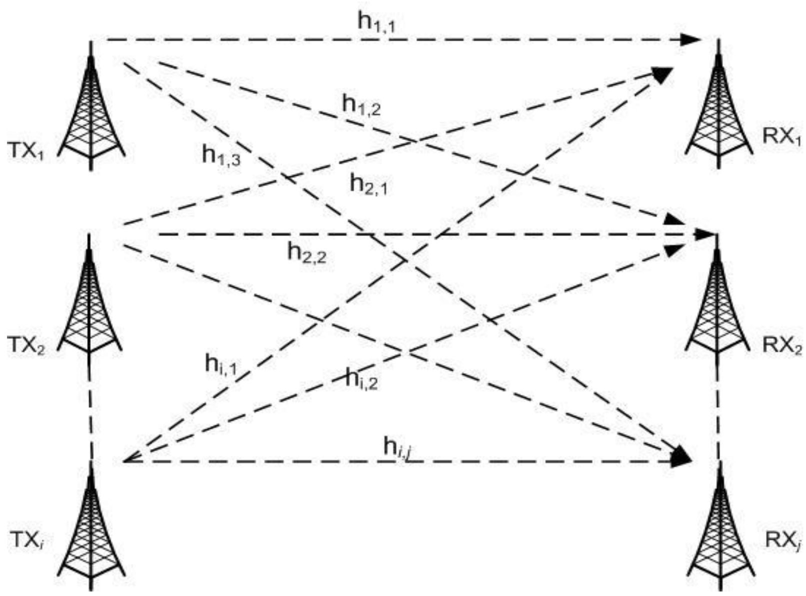

Figure 1. A generalized MIMO wireless communication system.

The main idea behind MIMO is that, the sampled signals in spatial domain at both the transmitter and receiver end are combined so that they form effective multiple parallel spatial data streams which increase the data rate. The occurrence of diversity also improves the quality that is the bit-error rate (BER) of the communication. [2]

2.2 Single Input Single Output versus MIMO channel capacity

In communication systems, input discrete source symbols are mapped into a sequence of channel symbols which are then transmitted through the wireless channel. The transmission of channel symbols through the wireless channel is by nature random and random noise is added to the channel symbols. The measure of how much information that can be transmitted and received with minimum probability of error is called the channel capacity. [3]

A Single Input Single Output system involves the use of one antenna both at the transmitter and receiver end. To a telecommunications engineer, there exit a limit at which reliable transmission of information is not possible for a given transmission bandwidth and power. These limits where discovered by Claude Shannon in 1948 when he established the principles of information and communication theory on his various publications. Shannon also established the conditions that enable the transmission of information over a noisy channel at a given rate, for a given power of the signal and noise. [4]

These limiting factors are the finite bandwidth and the *S/N* of the channel. This is because for a communication channel to accommodate the signal spectrum, enough transmission bandwidth is needed otherwise there will be distortion. ”The higher data rate is to be transmitted, the shorter digital pulses must be used and the shorter digital pulses are used for transmission, the wider bandwidth is required”. [5]

For a deterministic channel with a bandwidth (B) with additive noise, Shannon proved that information with a rate of *r* bits per second (bps) can be transmitted with a small error probability provided that the bit rate is less than the capacity of the channel $r < C$. The Shannon formulae that can be applied to determine the maximum capacity *C* of the channel is of the form

$$C = B\log_2[1 + S/N]\ [(\mathrm{bits/s})/\mathrm{H_Z}] \qquad (2.2.1)$$

where *S/N*, the signal-to-noise ratio and *B* the bandwidth of the transmission channel. [4]

Equation (2.2.1) informs us of how power and bandwidth are related. Assuming we have a channel with additive noise *N* and that we have some freedom of choosing the average transmission power *S*, to set up a reliable transmission link to send r bits per second. From the Shannon theorem, the data rate *r* cannot exceed capacity *C*, $r < C$ as in equation (2.2.1), but we still have one degree of freedom in the choice of bandwidth *B* and power *S*. It can be realized that, for a given signal-to-noise ratio *S/N*, if we wish to double *C*, we have to double the bandwidth *B*. On the other hand, if we double *C*, for a given *B* we have to evaluate the *S/N*. [4]

The main importance of MIMO channel technique is to improve the capacity of the channel and therefore it is important to compare the capacity of a SISO system to MIMO system. In SISO a system, the Shannon formula in equation (2.2.1) can be applied to determine the capacity of the system. However, for a precise comparison, it is important that the MIMO system is transmitting with a power the same as that of a SISO system. Therefore if the power radiated by a SISO system is P_s (SNR), then the power radiated from each antenna of a MIMO system with *N*TX transmit antennas must be P_s/N. [4]

Hence, for a MIMO system with *N*TX and *M*RX antennas using diversity at transmitter and receiver end, the capacity of the system can be determined by the formula

$$C = B\log_2[\mathbf{det}(\mathbf{I}_M + \frac{P_s}{N}\,\mathbf{HH}^*)] \text{ bps} \qquad [4]\ (2.2.2)$$

Where (*) means transpose-conjugate of $\mathbf{H}$ and $\mathbf{H}$ is the $M \times N$ channel matrix. $\mathbf{I}_M$ indicates the identity matrix of dimension $N \times M$, in this case $M = N$ =2 or more. [1] However, since signals transmitted over MIMO channel have to be linearly independent and orthogonal, interference averages to zero. Hence from equation (2.2.2) it can be seen that if the signal power P_s and the noise level N are the same then the more multiple antennas are used at the receiver, the more power is collected increasing the channel capacity and bandwidth.[10]

Example 2.1

Suppose that the spectrum of a SISO communication channel is 1MHz and the signal-to-noise ratio is 24dB. Using Shannon formula in equation (2.2.1)

$$C = B\log_2[1 + S/N]\ [(\text{bits/s})/\text{H}_\text{Z}]$$

where C is the capacity, B the bandwidth of the channel and S/N the signal-to-noise ratio.

By definition

$$S/N_{dB} = 10\log_{10}(S/N).$$

Therefore

$$24 = 10\log_{10}(S/N)$$

$$2.4 = \log_{10}(S/N).$$

Applying the inverse function for the log function, the exponential function base 10 to both sides is expressed as

$$S/N = 10^{2.4} = 251.$$

Hence the capacity is calculated as

$$C = 1\text{MHz}\log_2(1 + 251) = 10^6 \log_2(252)$$

We now use the change of base formula to log to base 2 as

$$\log_2(a) = \frac{\log_{10}(a)}{\log_{10}(2)}$$

$$C = 10^6 . \frac{\log_{10}(252)}{\log_{10}(2)} \quad = 10^6 \text{x } 7.977 \approx 8\text{Mbps}$$

However, if we increase the number of antennas at both transmits and receive end of the SISO system to 2 and apply the MIMO channel capacity formula in equation (2.2.2) to the 2x2 MIMO system, with the same channel bandwidth of 1MHz and signals-to-noise ratio of 24dB.

Example 2.2

Having a channel matrix

$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and its conjugate transpose as $\mathbf{H}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then the capacity of the 2 x 2 MIMO channel is calculated using

$$C = B\log_2[\mathbf{det}(\mathbf{I}_M + \frac{P_S}{N}\mathbf{HH}^*)] \text{ bps}$$

Where $\mathbf{I}_M$ is a 2 x 2 identity matrix and $P_s/N = (S/N)$ signal-to-noise ratio, the capacity is calculated as

$$C = 10^6 \log_2\left[\mathbf{det}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 251 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right]$$

$$C = 10^6 \log_2\left[\mathbf{det}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 251 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right]$$

$$C = 10^6 \log_2\left[\mathbf{det}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 251 & 0 \\ 0 & 251 \end{pmatrix}\right)\right]$$

$$C = 10^6 \log_2\left[\mathbf{det}\begin{pmatrix} 252 & 0 \\ 0 & 252 \end{pmatrix}\right]$$

$$C = 10^6 \log_2[252(252) - 0] = 10^6 \log_2 63504$$

Applying the change of base formula,

$$C = 10^6 . \frac{\log_{10}(63504)}{\log_{10}(2)} \quad = 10^6 \text{x } 15.955 \approx 16\text{Mbps}$$

Comparing the capacities of example (2.1) and (2.2) has proven that the capacity of the SISO channel can be doubled or increased by a factor of 2 if the number of antennas at both transmitter and receiver end of the SISO channel are increased to 2.

3 Representation of MIMO channel

3.1 A 2 x 2 MIMO channel model

The first channel model to be considered in this project will be a 2 x 2 MIMO system that is a system with 2 transmits (TX) and 2 receive (RX) antennas where different independent data streams are transmitted from multiple antennas to multiple receive antennas. This channel model will be extended to a 3 x 3 MIMO system and even more to illustrate the channel characteristics in relation to the increase in the number of antennas. The signals considered in the MIMO systems of this project are baseband signals ignoring modulation processes and concentrating on the up and down frequency conversion. Therefore the signals on the *i-th* transmit antenna will be denoted x_i while the received signal on the *j-th* receive antenna denoted as y_j. [9] Figure 2 shows the antenna set-up and the various unknown channel coefficients.

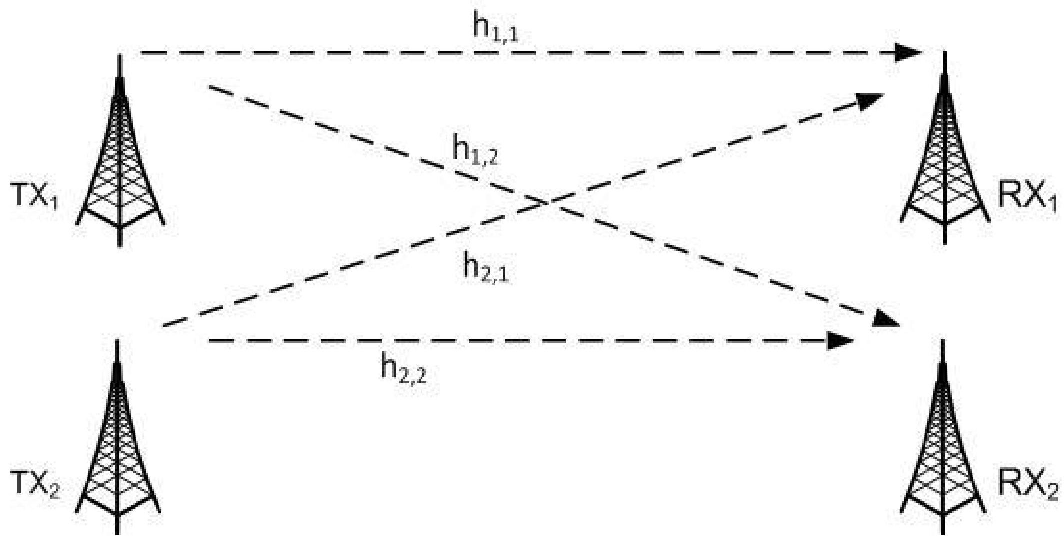

Figure 2. Channel characteristic of a 2 x 2 MIMO wireless communication system.

Since the coefficient of the unknown in the channel matrix $\mathbf{W_c}$ and the number of transmitted signal $\mathbf{X}$ is equal to the number of received signal $\mathbf{Y}$, the equation can be solved if the channel $\mathbf{W_c}$ is inversed which in this case a 2 x 2 matrix inversion.

3.2 Operational principles of a MIMO system

To derive the channel characteristics, MIMO system transmits specified and known training signals regularly from all transmitters of the system and these transmitted signals are received at the receiver. Based on the received signals, the receiver calculates the characteristics of all channel paths from each transmitted antenna to each receiving antenna. In order to prove that MIMO work, the transmitted signal $\mathbf{X}$ has to be solved from the group of equations in equation (3.2.1) below. We also assume that the system is noise free and line of sight (LOS). Reference to figure 3 below, if the transmitted signal is represented to be $\mathbf{X}$ and the received signal $\mathbf{Y}$. If the channel characteristics matrix is $\mathbf{W}_c$, we may write

$$\mathbf{Y}=\mathbf{X}\ \mathbf{W}_c \tag{3.2.1}$$

If the channel matrix has *N* rows as many as there are transmitting antennas with index i. Then transmitted signal vector is written as

$$\mathbf{X} = [\mathbf{x}_1,\ \mathbf{x}_2, \ldots \mathbf{x}_N] \tag{3.2.2}$$

Also if the channel matrix has *M* columns, as there are receiving antennas with index j. Then the received signal vector is

$$\mathbf{Y} = [\mathbf{y}_1,\ \mathbf{y}_2, \ldots \mathbf{y}_M] \tag{3.2.3}$$

These vectors are extended later to matrixes by inserting *K* samples into each column. The channel matrix contains path characteristics $h_{i,j}$ as

$$\mathbf{W}_c = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,M} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N,1} & h_{N,2} & \cdots & h_{N,M} \end{pmatrix}$$

Example 3.1 explains how independent transmitted signals can be transmitted from multiple transmitting antennas to multiple receiving antennas when channel characteristics are known. We should be able to calculate for transmitted signals if the received signal and the channel matrix are known.

Example 3.1

Let us take a MIMO system with *M* = *N* = 2 as in figure 3. We need to solve the transmitted signal when the received signals and channel are known. The channel matrix is an *N* (rows) by *M* (columns) matrix and the first index in each matrix element stands for row (transmitting antenna) and the second for column (receiving antenna).

Now if $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2)$ and $\mathbf{W}_c = \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}$

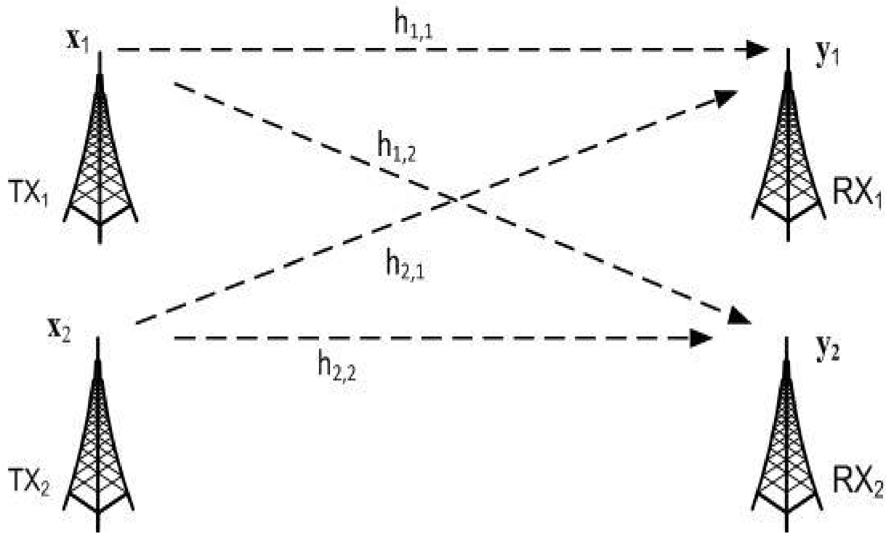

Figure 3. A 2 x 2 MIMO system with channel characteristics.

Figure 3 illustrate a 2 x 2 MIMO system showing the transmitted signals, the channel characteristics and the received signals.

Given $\mathbf{Y} = \mathbf{X}\,\mathbf{W}_c$ then

$$(\mathbf{y}_1, \mathbf{y}_2) = (\mathbf{x}_1, \mathbf{x}_2) \times \begin{pmatrix} \mathrm{h}_{1,1} & \mathrm{h}_{1,2} \\ \mathrm{h}_{2,1} & \mathrm{h}_{2,2} \end{pmatrix} \quad (3.2.4)$$

The solution of $\mathbf{X}$ can deduce from equation (3.2.4) as

$$\mathbf{y}_1 = \mathbf{x}_1 \mathrm{h}_{1,1} + \mathbf{x}_2 \mathrm{h}_{2,1} \quad (3.2.5)$$

$$\mathbf{y}_2 = \mathbf{x}_1 \mathrm{h}_{1,2} + \mathbf{x}_2 \mathrm{h}_{2,2} \quad (3.2.6)$$

This implies that from equation (3.2.5)

$$\mathbf{x}_2 = \frac{\mathbf{y}_1 - \mathbf{x}_1 \mathrm{h}_{1,1}}{\mathrm{h}_{2,1}} \quad (3.2.7)$$

Substituting equation (3.2.7) into equation (3.2.6), we have

$$\mathbf{y}_2 = \mathbf{x}_1 \mathrm{h}_{1,2} + \left(\frac{\mathbf{y}_1 - \mathbf{x}_1 \mathrm{h}_{1,1}}{\mathrm{h}_{2,1}}\right) \mathrm{h}_{2,2}$$

$$\mathbf{y}_2 = \mathbf{x}_1 \mathrm{h}_{1,2} + \frac{\mathrm{h}_{2,2}}{\mathrm{h}_{2,1}}\left(\mathbf{y}_1 - \mathbf{x}_1 \mathrm{h}_{1,1}\right) = \frac{\mathrm{h}_{2,2}}{\mathrm{h}_{2,1}}\mathbf{y}_1 + \left(\mathrm{h}_{1,2} - \frac{\mathrm{h}_{1,1}\mathrm{h}_{2,2}}{\mathrm{h}_{2,1}}\right)\mathbf{x}_1$$

$$\mathbf{x}_1 = \frac{\mathbf{y}_2 - \dfrac{h_{2,2}}{h_{2,1}}\mathbf{y}_1}{h_{1,2} - \dfrac{h_{1,1}h_{2,2}}{h_{2,1}}} = \frac{\dfrac{\mathbf{y}_2 h_{2,1} - h_{2,2}\mathbf{y}_1}{h_{2,1}}}{\dfrac{h_{1,2}h_{2,1} - h_{1,1}h_{2,2}}{h_{2,1}}}$$

$$\mathbf{x}_1 = \frac{\mathbf{y}_2 h_{2,1} - h_{2,2}\mathbf{y}_1}{h_{2,1}} \times \frac{h_{2,1}}{h_{1,2}h_{2,1} - h_{1,1}h_{2,2}}$$

$$\mathbf{x}_1 = \frac{\mathbf{y}_2 h_{2,1} - \mathbf{y}_1 h_{2,2}}{h_{1,2}h_{2,1} - h_{1,1}h_{2,2}} \quad (3.2.8)$$

Applying the same process, from equation (3.2.5) the second transmitted signal can be calculated as

$$\mathbf{x}_1 = \frac{\mathbf{y}_1 - \mathbf{x}_2 h_{2,1}}{h_{1,1}}$$

$$\mathbf{y}_2 = \frac{h_{1,2}}{h_{1,1}}\left(\mathbf{y}_1 - \mathbf{x}_1 h_{2,1}\right) + \mathbf{x}_2 h_{2,2}$$

$$\mathbf{y}_2 = \frac{h_{1,2}}{h_{1,1}}\mathbf{y}_1 - \frac{h_{2,1}h_{1,2}}{h_{1,1}}\mathbf{x}_2 + \mathbf{x}_2 h_{2,2} = \frac{h_{1,2}}{h_{1,1}}\mathbf{y}_1 + \left(h_{2,2} - \frac{h_{2,1}h_{1,2}}{h_{1,1}}\right)\mathbf{x}_2$$

$$\mathbf{x}_2 = \frac{\mathbf{y}_2 - \frac{h_{1,2}}{h_{1,1}}\mathbf{y}_1}{h_{2,2} - \frac{h_{2,1}h_{1,2}}{h_{1,1}}} = \frac{\frac{\mathbf{y}_2 h_{1,1} - h_{1,2}\mathbf{y}_1}{h_{1,1}}}{\frac{h_{2,2}h_{1,1} - h_{2,1}h_{1,2}}{h_{1,1}}}$$

$$\mathbf{x}_2 = \frac{\mathbf{y}_2 h_{1,1} - h_{1,2}\mathbf{y}_1}{h_{1,1}} \times \frac{h_{1,1}}{h_{2,2}h_{1,1} - h_{2,1}h_{1,2}}$$

$$\mathbf{x}_2 = \frac{\mathbf{y}_2 h_{1,1} - \mathbf{y}_1 h_{1,2}}{h_{2,2}h_{1,1} - h_{2,1}h_{1,2}}$$

Hence the solution for the transmitted signal $\mathbf{X}$ from equation (3.2.4) is given in a vector form as

$$\mathbf{X} = \left(\frac{y_2 h_{2,1} - y_1 h_{2,2}}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}}, \frac{y_2 h_{1,1} - y_1 h_{2,1}}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}}\right) \quad (3.2.9)$$

We can see from example 3.1 that the transmitted signal $\mathbf{X}$ can be determined if the channel characteristics $\mathbf{W_c}$ and the received signal $\mathbf{Y}$ are known. We can also solve for the transmitted signal $\mathbf{X}$ using matrix representation if the channel matrix $\mathbf{W_c}$ is inversed and the received signal $\mathbf{Y}$ also known. The next example explains how the transmitted signals $\mathbf{X}$ can be determined if the channel matrix $\mathbf{W_c}$ is inversed and the received signal $\mathbf{Y}$ is known.

Example 3.2

Reference to equation (3.2.1) and equation (3.2.4), if the groups of equations are given as

$$\mathbf{Y} = \mathbf{X}\,\mathbf{W}_c,\ (\mathbf{y}_1, \mathbf{y}_2) = (\mathbf{x}_1, \mathbf{x}_2) \times \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}$$

Then the transmitted signal $\mathbf{X}$ can be solve if we invert the channel matrix and multiply it with the received signal $\mathbf{Y}$.

$$\mathbf{X} = \mathbf{W}_c^{-1}\,\mathbf{Y} \qquad (3.2.10)$$

The inverse of the channel matrix $\mathbf{W}_c^{-1}$ can be determined by first finding the adjoint of the channel matrix and then dividing it with its determinant. Mathematically, the inverse of the channel matrix can be represented as

$$\mathbf{W}_c^{-1} = \frac{\mathbf{adj}(\mathbf{W}c)}{\mathbf{det}(\mathbf{W}c)} \qquad (3.2.11$$

Where $\mathbf{adj(W_c)}$ is the adjoint of the channel matrix that is formed by taking the transpose of the cofactor matrix of $\mathbf{W}_c$. Since this is a 2 x 2 matrix, the cofactor of the channel matrix is calculated as

Cofactor matrix of $\mathbf{W}_c = \begin{pmatrix} h_{2,2} & -h_{1,2} \\ -h_{2,1} & h_{1,1} \end{pmatrix}$

The adjoint of the channel matrix $\mathbf{W}_c$ is also obtained as

$$\mathbf{adj}(\mathbf{W}_c) = \begin{pmatrix} h_{2,2} & -h_{1,2} \\ -h_{2,1} & h_{1,1} \end{pmatrix}$$

The determinant of the channel matrix $\mathbf{W}_c$ is also deduced from the expression

$$\mathbf{det}(\mathbf{W}_c) = h_{1,1}h_{2,2} - h_{2,1}h_{1,2}$$

Hence the inverse of the channel matrix $\mathbf{W}_c$ is expressed as

$$\mathbf{W}_c^{-1} = \frac{1}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}} \begin{pmatrix} h_{2,2} & -h_{1,2} \\ -h_{2,1} & h_{1,1} \end{pmatrix}$$

From equation (3.2.10) the transmitted signals can be determined as

$$[\mathbf{x}_1, \mathbf{x}_2] = \frac{1}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}} \begin{pmatrix} h_{2,2} & -h_{1,2} \\ -h_{2,1} & h_{1,1} \end{pmatrix} [\mathbf{y}_1, \mathbf{y}_2]$$

$$\mathbf{x}_1 = \frac{\mathbf{y}_1 h_{2,2} - \mathbf{y}_2 h_{1,2}}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}}$$

$$\mathbf{x}_2 = \frac{\mathbf{y}_2 h_{1,1} - \mathbf{y}_1 h_{2,1}}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}}$$

The transmitted signals are now represented in a vector form as

$$\mathbf{X} = \left(\frac{\mathbf{y}_1 h_{2,2} - \mathbf{y}_2 h_{1,2}}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}} , \frac{\mathbf{y}_2 h_{1,1} - \mathbf{y}_1 h_{2,1}}{h_{1,1}h_{2,2} - h_{2,1}h_{1,2}} \right) \qquad (3.1.12)$$

Hence it is possible to solve the transmitted signals with group of equations (3.2.4) and also with the help of the matrix representation (equation 3.2.10). Therefore it can be proven that the transmitted signals can be determined if the channel matrix and received signals are known. This explains that MIMO works, but to illustrate its practical operation the analyses have to continue. The next chapter explains in details the principles or steps required to present this project with practical examples. It should be noted that the analyses of this are based on discrete MIMO system.

4 Channel estimation procedure

4.1 Channel characteristics estimation

In order to estimate the channel characteristics, we expand each transmitted and received signals in time and write into signal matrix columns *K* discrete samples in time. The signals matrixes get *K* rows and as many columns as we have antennas, *N* or *M*. In general a MIMO system involves multiple antennas at the transmitting end and multiple receivers at the receiving end. Figure 4 show a general representation of a 3 x 3 general MIMO system.

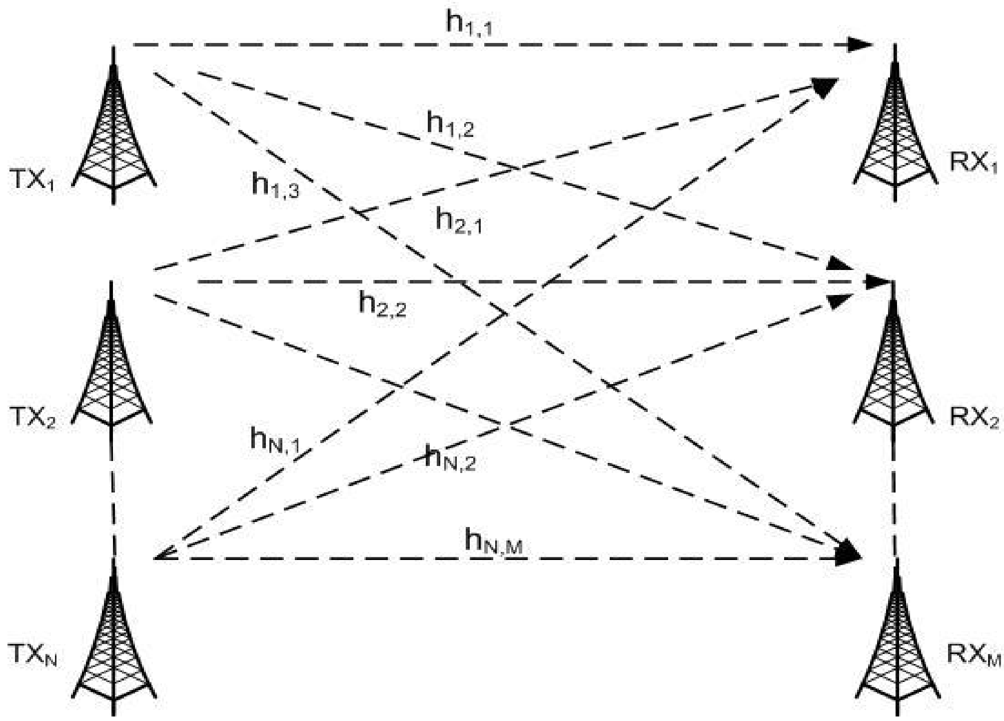

Figure 4. General 3 x 3 MIMO system with unknown channel characteristics.

In order to determine the characteristic of the channel, both the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$ have to be known. If the transmitted and received signals are of the form

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N] = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{K,1} & \cdots & x_{K,N} \end{pmatrix} \tag{4.1.1}$$

$$\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots \mathbf{y}_M] = \begin{pmatrix} y_{1,1} & \cdots & y_{1,M} \\ \vdots & \ddots & \vdots \\ y_{K,1} & \cdots & y_{K,M} \end{pmatrix} \tag{4.1.2}$$

If the channel transfer matrix $\mathbf{W_c}$ can be determine, then it means the transmitted signals can also be determine because the received signal $\mathbf{Y}$ is known. We may write the expression between these vector signals as

$$\mathbf{Y} = \mathbf{X}\,\mathbf{W_c} \tag{4.1.3}$$

In order to determine the transmitted signals, the channel transfer matrix, $\mathbf{W_c}^{-1}$ have to be inverted and then multiplied with the received signal matrix $\mathbf{Y}$.

$$\mathbf{X} = \mathbf{Y}\,\mathbf{W}_{\mathrm{c}}^{-1}$$

The channel transfer matrix that we have to solve with the help of known training signal $\mathbf{X}$ and the received signal $\mathbf{Y}$ has the form

$$\mathbf{W}_K = \begin{pmatrix} w_{1,1} & \cdots & w_{1,K} \\ \vdots & \ddots & \vdots \\ w_{K,1} & \cdots & w_{K,K} \end{pmatrix} \tag{4.1.4}$$

The channel transfer matrix is calculated periodically with the help of the known training signals and remains constant over information transmission time. It is then recalculated when new information is being transmitted. The channel characteristics in equation (4.1.4) are defined for each signal path at discrete time instants 1,2,...K. However, we need to derive $\mathbf{W}_{\mathrm{c}}$ and this can be derived with the help of the transmitted signals $\mathbf{X}$ and the corresponding known received signals $\mathbf{Y}$ measured at the receiver. Equation (4.1.4) expresses the channel matrix $\mathbf{W_c}$ which do not need to be a square matrix. Because, for example if we have a 4x4 MIMO system with 100 samples, we do not need to have a 100x100 channel matrix. However, 5 samples can be transmitted from the four (4) antennas at a time. The next section explains the procedures required in order to estimate the channel transfer matrix $\hat{\mathbf{W}}_{\mathbf{c}}$.

4.2 Channel identification algorithm

To model the transfer channel a common space matrix (orthonormal basis matrix $\mathbf{U_x}$) is first generated and then used to map both the transmitted and received signals that vary in space (multiple antennas) and in time (samples in time). This common orthonormal basis matrix is obtained by decomposing either the transmitted signal $\mathbf{X}$ or the received signal $\mathbf{Y}$. Hence, MIMO channel problem can be solved using four step approach under a condition where there is no noise ($\mathbf{N} = \mathbf{0}$) and if the transmitted signals are orthogonal. These steps are summarized as follows:

(1) Finding an orthonormal basis $\mathbf{U_x}$ of the transmitted signal matrix $\mathbf{X}$ using the Gram-Schmidt procedure. [9, 166]

(2) In the *K*-dimensional signal vector space spanned by $\mathbf{U_x}$, we express the *N* column vectors of the transmitted signal $\mathbf{X}$ by the projection onto the orthogonal axes of $\mathbf{U_x}$. [9, 166]

$$\mathbf{W_x} = \mathbf{U_x}\,\mathbf{X} \tag{4.2.1}$$

(3) In the K-dimensional signal vector space spanned by $\mathbf{U_x}$, we express the M column vectors of the received signal matrix $\mathbf{Y}$ by their projection onto the orthogonal axes of $\mathbf{U_x}$. [9, 166]

$$\mathbf{W_y} = \mathbf{U_x}\,\mathbf{Y} \tag{4.2.2}$$

(4) Calculate the inverse or the pseudo inverse of the Fourier coefficients of the transmitted signal $\mathbf{W_x}$ and find an estimate of channel transfer matrix $\acute{\mathbf{W}}_\mathbf{c}$. [9, 166]

$$\hat{\mathbf{W}}_\mathbf{c} = \mathbf{W}_x^{-1}\,\mathbf{W}_y \text{ or } \hat{\mathbf{W}}_\mathbf{c} = \mathbf{W}_x^{+}\,\mathbf{W}_y$$

Figure 5 shows the various steps required to model the transfer channel $\hat{\mathbf{W}}_\mathbf{c}$.

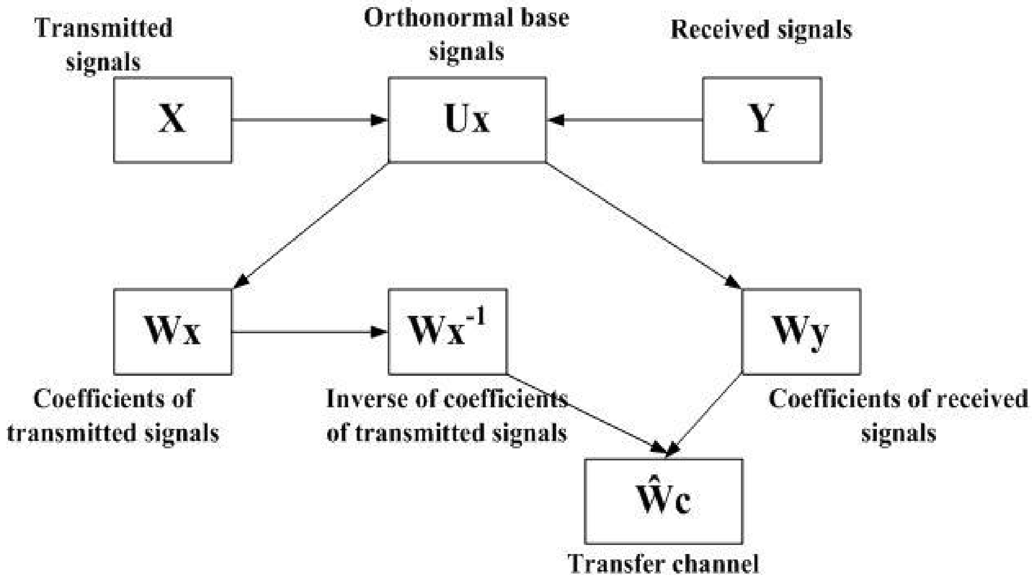

Figure 5. Transfer channel estimation processes.

Reference to figure 5, it can deduce that the first principle is to map both the transmitted and the received signals with a common space matrix, the orthonormal basis vector. The second process in figure 5 is to determine the Fourier coefficients of both the transmitted and the received signals. The final process is to model the transfer channel with the help of the Fourier confidents obtained. It is important to explain in details the four listed steps for clear understanding. The next section explains these four principles.

4.2.1 Orthonormal basis $\mathbf{U}_x$ estimation

To be able to estimate the transfer channel matrix $\hat{\mathbf{W}}_c$, a common space matrix is required to map both the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$ together. The orthonormal base matrix $\mathbf{U}_x$ serves as the common space matrix needed to map the transmitted signal $\mathbf{X}$ to the received signal $\mathbf{Y}$. It is important to know that, the orthonormal basis needed for the mapping can be derived either using the transmitted signal $\mathbf{X}$ or the received signal $\mathbf{Y}$. Hence, the orthonormal basis $\mathbf{U}_x$ of the transmitted signal $\mathbf{X}$ is calculated by taking the matrix $\mathbf{U}_x$ obtained from the decomposition of transmitted signal $\mathbf{X}$.

In linear algebra, a matrix such as the transmitted signal $\mathbf{X}$ can be decomposition into the product $\mathbf{X} = \mathbf{U}_x \mathbf{R}$ where $\mathbf{U}_x$ is an orthogonal matrix in this case the orthonormal basis and $\mathbf{R}$ an upper triangular matrix. It should be noted that, the format of the matrix $\mathbf{X}$ is K-by-N while the size of the orthonormal base matrix $\mathbf{U}_x$ is always K-by-K. [9,166] Gram-Schmidt procedure is one way to decompose a column rank matrix and this procedure will be used in this project. There are also other methods of decomposing a matrix such as QR-decomposition, LU decomposition, Cholesky decomposition, etc. See appendix 1 for Gram-Schmidt process.

4.2.2 Fourier coefficients of transmitted signal $\mathbf{W}_x$

After determining the orthonormal basis $\mathbf{U}_x$, we have to map the known transmitted signal $\mathbf{X}$ to the orthonormal space matrix $\mathbf{U}_x$ and this is done by calculating the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_x$. The mapping of the transmitted signal $\mathbf{X}$ with respect to the orthonormal basis $\mathbf{U}_x$ is expressed

$$\mathbf{W}_x = \mathbf{U}_x \mathbf{X} \qquad (4.2.1)$$

The generalized Fourier coefficients are coefficients of any orthogonal set of functions over which signals are split up. Therefore the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_x$ tell us how much each column (signal) of the transmitted signal $\mathbf{X}$ contains each orthogonal column component in the orthonormal base matrix $\mathbf{U}_x$. In this case we are splitting the transmitted signal with the help of the orthonormal basis $\mathbf{U}_x$. It should be noted that the multiplication of $\mathbf{U}_x \mathbf{U}_x^T = \mathbf{I}$ where $\mathbf{I}$ is a K-by-K identity matrix and size of $\mathbf{W}_x$ is K-by-N. [9,166]

4.2.3 Fourier coefficients of training (received) signal $\mathbf{W}_y$

The received signal $\mathbf{Y}$ is also mapped to the orthonormal space matrix $\mathbf{U}_x$ by multiplying the received signal $\mathbf{Y}$ with the orthonormal base matrix $\mathbf{U}_x$. This means that we have to split the received signal $\mathbf{Y}$ with the help of the orthonormal basis $\mathbf{U}_x$. Hence, the calculation the Fourier coefficients of the received signal as

$$\mathbf{W}_y = \mathbf{U}_x \mathbf{Y} \qquad (4.2.2)$$

Where the size of $\mathbf{W}_y$ is K-by-M.

4.2.4 Transfer channel matrix estimation $\hat{\mathbf{W}}_c$

Finally, we derive the most favorable estimate of the channel transfer matrix from the expression $\hat{\mathbf{W}}_c = \mathbf{W}_x^{-1} \mathbf{W}_y$. If the Fourier coefficient of the transmitted signal $\mathbf{W}_x$ is not a square matrix then we use the expression $\hat{\mathbf{W}}_c = \mathbf{W}_x^{+} \mathbf{W}_y$ where $\mathbf{W}_x^{+}$ the pseudo inverse of $\mathbf{W}_x$. [9,166] .This follows from the formula

$$\mathbf{Y} = \mathbf{X} \hat{\mathbf{W}}_c$$

$$\hat{\mathbf{W}}_c = \mathbf{X}^{-1} \mathbf{Y} = \mathbf{W}_x^{-1} \mathbf{U}_x \mathbf{Y} = \mathbf{W}_x^{-1} \mathbf{U}_x \mathbf{U}_x^{-1} \mathbf{W}_y = \mathbf{W}_x^{-1} \mathbf{W}_y \qquad (4.2.3)$$

From equation (4.2.3), again it can be seen that to estimate the channel transfer matrix $\hat{\mathbf{W}}_c$, the inverse of the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_x$ is needed since it is a square matrix. However, in the case where $\mathbf{W}_x$ is not a square matrix, the pseudo inverse of $\mathbf{W}_x$ is calculated. [9,166]

The orthonormal basis $\mathbf{U}_x$ plays an important role in determining the channel characteristics and therefore it is important to develop an algorithm to generate $\mathbf{U}_x$. Not only to generate the orthonormal basis matrix but to illustrate the entire matrixes required to estimate the transfer channel. Hence, Microsoft Excel® is used to develop an algorithm for Gram-Schmidt procedure to generate $\mathbf{U}_x$, the Fourier coefficients of both transmitted and received signals as well as the transfer channel. This algorithm can be accessed from a compact disk (CD) attached to this project. The next chapter explains with an example how to determine the orthonormal basis $\mathbf{U}_x$ using Gram-Schmidt procedure by decomposing the transmitted signal $\mathbf{X}$ and applying this procedure to two different scenarios. First the calculation of the orthonormal basis $\mathbf{U}_x$ in the case where the transmitted signal $\mathbf{X}$ is a square matrix and in the second case where the transmitted signals $\mathbf{X}$ is not a square matrix.

5 Orthonormal space concept

5.1 Orthonormal basis of transmitted signal

Orthonormal basis is a coordination system where we can present as many dimensions as is the maximum number of antennas of the transmitted and received signals. The main reason why we need the orthonormal base is to have a common coordinated system in order to combine the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$. The number of dimension of the orthonormal base matrix depends on the maximum number of antennas at both the transmitted and the receiver end of the MIMO system. It does not actually matter what kind of signal ($\mathbf{X}$ or $\mathbf{Y}$) used in generating the orthonormal space but it is most convenient to use the transmitted training signal $\mathbf{X}$ which is defined to contain linearly independent column signals.

Suppose K = 3 time samples per transmitted signal $\mathbf{X}$ and per received signal $\mathbf{Y}$ respectively of a 3 x 3 channel transmission system were observed. Figure 6 shows a Line of Sight (LOS) MIMO system made up of three transmitters and three receivers.

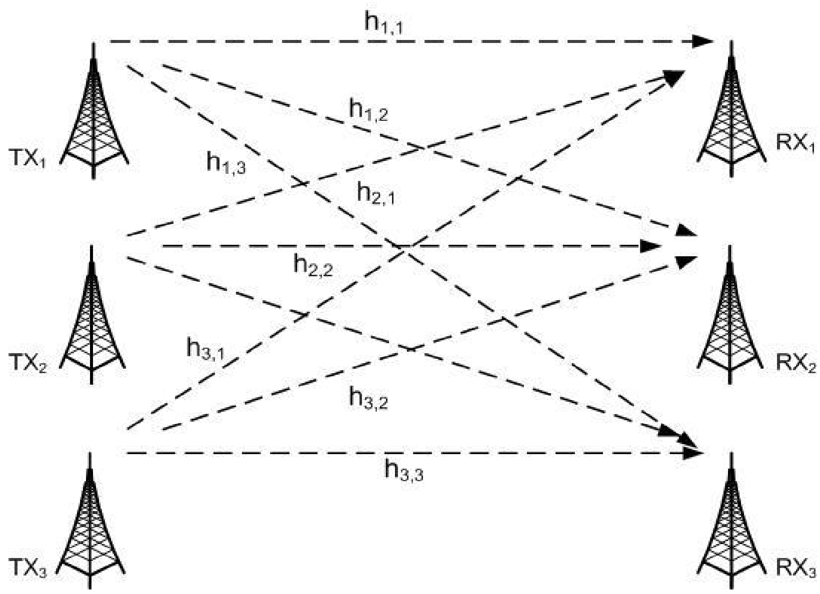

Figure 6. A 3 x 3 MIMO wireless communication system.

In figure 6, N = 3 transmitted signals and M = 3 received signals, the format of the unknown channel transfer matrix $\hat{\mathbf{W}}_c$ would be 3 x 3. Assume that all signals are real valued and the transmitted signals and received signals are observed as signal matrices. Then we need the orthonormal basis matrix $\mathbf{U}_x$ to solve for an estimated channel transfer matrix $\hat{\mathbf{W}}_c$. It is easier to calculate the orthonormal base matrix $\mathbf{U}_x$ of a square matrix; therefore in the next section we estimate the channel model by calculating first the orthonormal basis matrix $\mathbf{U_x}$ of a simple 2 x 2 MIMO system and then proceed to a more complex 3 x 3 MIMO system.

5.2 Orthonormal basis $\mathbf{U}_\mathrm{x}$ and transfer channel $\hat{\mathbf{W}}_\mathrm{c}$ estimation

5.2.1 A 2x2 MIMO system orthonormal base and channel estimation

This section explains how to calculate the orthonormal basis $\mathbf{U_x}$ of a 2 x 2 MIMO system with the help of the transmitted signal $\mathbf{X}$. To demonstrate the steps in subchapter 4.2, we will have a numerical example to explain how to generate the orthonormal basis $\mathbf{U_x}$ of a full rank square matrix using linearly independent transmitted signals in $\mathbf{X}$. The Gram-Schmidt procedure will be used in the decomposition of the transmitted signal $\mathbf{X}$. See appendix 1 for Gram-Schmidt procedure. The equation to decomposed in our calculation is of the form

$$\mathbf{X} = \mathbf{U_x R}; \quad \mathbf{R} = \mathbf{U_x}^{-1}\,\mathbf{X}. \qquad (5.2.1)$$

In equation (5.2.1) the transmitted signal $\mathbf{X}$ is divided into two components, $\mathbf{U}_\mathrm{x}$ the orthonormal basis and $\mathbf{R}$ the upper triangular matrixes. The upper triangular matrix $\mathbf{R}$ is calculated by first finding the inverse of the orthonormal basis matrix $\mathbf{U_x}$ and multiplying it with the transmitted signal $\mathbf{X}$. However, the upper triangular matrix $\mathbf{R}$ will is not needed in our analysis. The next very simple example explains how Gram Schmidt procedure can be applied to generate orthonormal base $\mathbf{U}_\mathrm{x}$ of the transmitted signal $\mathbf{X}$.

Example 5.2.1

Suppose the transmitted signal $\mathbf{X}$ to be decomposed is

$$\mathbf{X} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},\ \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then the first column vector signal of the orthonormal base $\mathbf{U_x}$ is obtained by normalizing the first column vector signal $\mathbf{x}_1$ of the transmitted signal $\mathbf{X}$.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The second column vector signal $\mathbf{e}_2$ of the orthonormal base $\mathbf{U_x}$ is obtained by subtracting from the second column vector $\mathbf{x}_2$ its component on the first dimension which is a projection of $\mathbf{x}_2$ in direction $\mathbf{e}_1$. The projection of $\mathbf{x}_2$ is calculated by the expression

$$\mathbf{u}_2 = \mathbf{x}_2 - (\mathbf{e}_1^{\mathrm{T}} \cdot \mathbf{x}_2)\,\mathbf{e}_1$$

$$\mathbf{u}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \left(-\frac{1}{\sqrt{2}}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

The vector $\mathbf{u}_2$ is normalized by dividing its column vector values by its length which gives $\mathbf{e}_2$. The length of $\mathbf{u}_2$ is given as

$$|\mathbf{u}_2| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

The second column vector of the orthonormal base is

$$\mathbf{e}_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Hence the orthonormal base $\mathbf{U}_x$ of the transmitted signal $\mathbf{X}$ is

$$\mathbf{U}_x = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5.2.2)$$

These column vector signals of the orthonormal base $\mathbf{U}_x$ can be represented in figure 7 below.

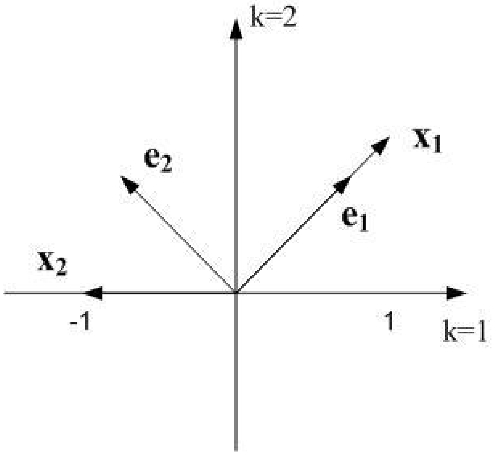

Figure 7. Vector representation of orthonormal base $\mathbf{U}_x$.

Figure 7 shows the various independent transmitted vector signals and the orthonormal base $\mathbf{U}_x$ column vectors obtained in equation (5.2.2).

To get a clear understanding of how the orthonormal base $\mathbf{U}_x$ is used in estimating the transfer channel $\hat{\mathbf{W}}_c$, simple examples explaining the general procedures listed in section 4.2 and the role of the orthonormal base $\mathbf{U}_x$ in the estimation of the transfer channel $\hat{\mathbf{W}}_c$ are illustrated. This example involves a MIMO system with two antennas at both transmits and received end of the system in two different scenarios. In both scenarios, the transmitted signals $\mathbf{X}$ are the same but different received signals.

Example 5.2.2

Suppose we have K = 2 time samples per transmitted and the received signals respectively in the system. If we assume that both the transmitted and the received signals in this scenarios are real valued observed as matrices

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (5.2.3)$$

As listed in chapter 4.2, we first have to calculate the orthonormal base $\mathbf{U_x}$ with the help of the transmitted signal $\mathbf{X}$ which will be used as a common signal space matrix to map the transmitted and the received signals together. The orthonormal base $\mathbf{U_x}$ is obtained using the Gram-Schmidt procedure. The first column of the orthonormal base $\mathbf{U_x}$ is obtained by normalizing the first column signal of the transmitted signal $\mathbf{X}$.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The second column vector signal $\mathbf{u}_2$ of the orthonormal base $\mathbf{U_x}$ is obtained by subtracting from the second column vector $\mathbf{x_2}$ its component on the first dimension which is a projection of $\mathbf{x_2}$ in direction $\mathbf{e}_1$. The projection of $\mathbf{x_2}$ is calculated by the expression

$$\mathbf{u}_2 = \mathbf{x}_2 - (\mathbf{e}_1^T \cdot \mathbf{x}_2)\, \mathbf{e}_1$$

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (1 \quad 0) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence $\mathbf{e}_2$ is obtained by normalizing $\mathbf{u}_2$ by diving its vector values with it length. The length of $\mathbf{u}_2$ is given as

$$|\mathbf{u}_2| = \sqrt{(0)^2 + (1)^2} = 1$$

The second column vector signal of the orthonormal base is

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The two dimensional orthonormal base $\mathbf{U_x}$ becomes

$$\mathbf{U_x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (5.2.4)$$

The next step is to map the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$ to the orthonormal base vector $\mathbf{U_x}$ obtained in equation (5.2.4). This is done by calculating their generalized Fourier coefficients respectively.

Example 5.2.3

The generalized Fourier coefficients of the transmitted signal $\mathbf{X}$ is calculated from the expression

$$\mathbf{W_x} = \mathbf{U_x}\,\mathbf{X}$$

$$\mathbf{W_x} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad (5.2.5)$$

Also the generalized Fourier coefficients of the received signals $\mathbf{Y}$ is calculated as

$$\mathbf{W_y} = \mathbf{U_x}\,\mathbf{Y}$$

$$\mathbf{W_y} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad (5.2.6)$$

The final step is to estimate the transfer channel using the expression

$$\hat{\mathbf{W}}_\mathbf{c} = \mathbf{W}_\mathrm{x}^{-1}\,\mathbf{W}_\mathrm{y}$$

This means that before the channel can be estimated, the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_\mathrm{x}$ have to be inverted and then multiplied with the generalized Fourier coefficients of the received signal $\mathbf{W}_\mathrm{y}$. The inverse of the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_\mathrm{x}$ which is a 2 x 2 matrix is calculated as

$$\mathbf{W}_\mathrm{x} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\mathbf{W}_\mathrm{x}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$$

$$\mathbf{W}_\mathrm{x}^{-1} = \frac{1}{1} \times \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad (5.2.7)$$

The transfer channel is estimated as

$$\hat{\mathbf{W}}_\mathbf{c} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\hat{\mathbf{W}}_\mathbf{c} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}w_{1,1} & w_{1,2}\\w_{2,1} & w_{2,2}\end{pmatrix} \qquad (5.2.8)$$

Figure 8 illustrates this simple example (5.2.3) showing how the transmitted signals are transformed by the channel to the receiver in the 2 x 2 MIMO system.

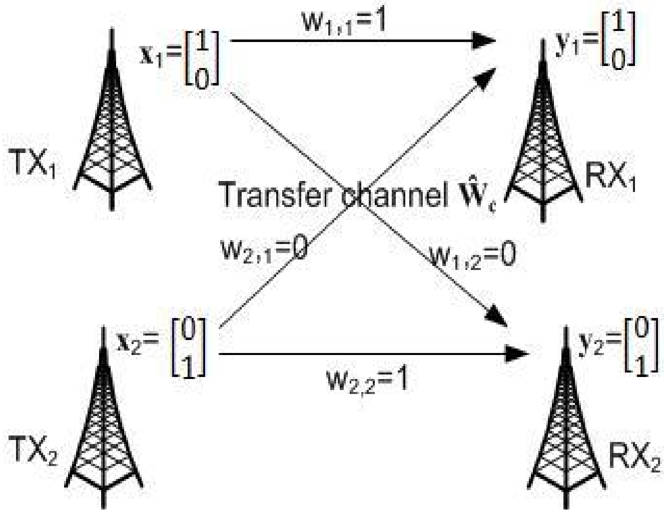

Figure 8. Example of a simple 2 x 2 MIMO system.

The graphs of all the steps (orthonormal base $\mathbf{U}_x$, Fourier coefficients of transmitted signal $\mathbf{W}_x$, Fourier coefficients of received signal $\mathbf{W}_y$ and the estimated transfer channel $\hat{\mathbf{W}}_c$) involved in the estimation of the transfer channel are shown in the 2 x 2 algorithm Microsoft Excel algorithm attached to this project.

In the second scenario, we will estimate the transfer channel in a case where the information transmitted is different from what was received. Not only will this enable to observe the characteristics of the transfer channel but helps us to estimate the real information transmitted.

Example 5.2.4

If the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$ observed are

$$\mathbf{X} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\ \mathbf{Y} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \tag{5.2.9}$$

As in example 5.2.2, we first have to find the orthonormal base $\mathbf{U}_x$ matrix, which will be used as a common space matrix to map the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$. However, since the transmitted signal $\mathbf{X}$ in this example is the same as the transmitted signal in example 5.2.2, the same orthonormal base matrix $\mathbf{U}_x$ (equation (5.2.4)) will be generated.

$$\mathbf{U}_x = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

The next step is to calculate the generalized Fourier coefficients of both the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$ respectively. Therefore generalized Fourier coefficients of the transmitted signal $\mathbf{X}$ is calculated as

$$\mathbf{W_x} = \mathbf{U_x}\,\mathbf{X}$$

$$\mathbf{W_x} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

The generalized Fourier coefficients of the received signal $\mathbf{Y}$ is also calculated as

$$\mathbf{W_y} = \mathbf{U_x}\,\mathbf{Y}$$

$$\mathbf{W_y} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \qquad (5.2.10)$$

The final step is to estimate the transfer channel by first finding the inverse of the Fourier coefficients of the transmitted signal $\mathbf{W}_x^{-1}$ and then multiplying it with the Fourier coefficients of the received signal $\mathbf{W}_y$ in the expression

$$\hat{\mathbf{W}}_\mathbf{c} = \mathbf{W}_x^{-1}\,\mathbf{W}_y$$

However, the generalized Fourier coefficient of the transmitted signal is the same as in equation (5.2.5) and therefore its inverse will produce the same results as in equation (5.2.7). Hence the inverse of the generalized Fourier coefficient of the transmitted signal is

$$\mathbf{W}_x^{-1} = \frac{1}{1} \times \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\mathbf{W}_x^{-1} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

The transfer channel is then estimated by the expression

$$\hat{\mathbf{W}}_\mathbf{c} = \mathbf{W}_x^{-1}\,\mathbf{W}_y$$

$$\hat{\mathbf{W}}_\mathbf{c} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \times \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

$$\hat{\mathbf{W}}_\mathbf{c} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \qquad (5.2.11)$$

Figure 9 shows how the transmitted signals of example 5.2.4 were transformed by the estimated transfer channel.

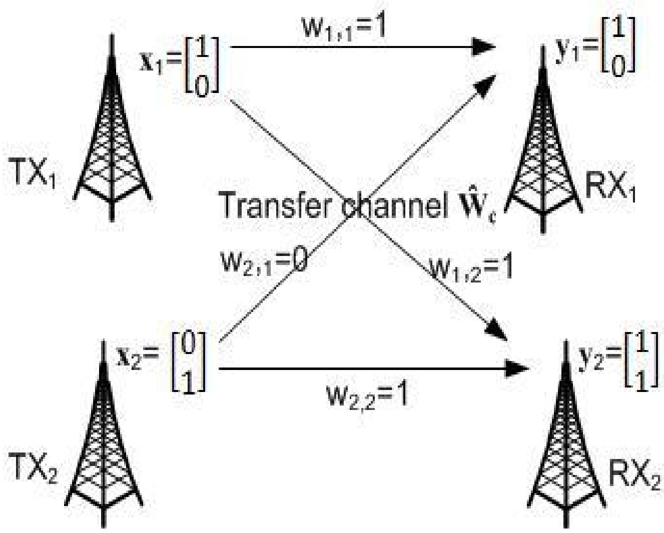

Figure 9. Example of 2 x 2 MIMO system.

The graphs of the various processes (orthonormal base $\mathbf{U}_x$, Fourier coefficient of transmitted signal $\mathbf{X}$, Fourier coefficient of received signal $\mathbf{X}$ and the estimated transfer channel $\hat{\mathbf{W}}_c$) involved in the estimation of the transfer channel in equation (5.2.11) will be produced if the received signals in the 2 x 2 algorithm is changed to that in equation (5.2.9). The next important step is also to estimate the received signal with help of the estimated channel $\hat{\mathbf{W}}_c$. The next section explains how to estimate the received signal when the transmitted signal and the transfer channel are known.

5.2.2 Received signal $\hat{\mathbf{Y}}$ when transmitted signal and channel are known

The transmitted known training signals are used in modelling the transfer channel $\hat{\mathbf{W}}_c$ as shown in examples (5.2.3) and (5.2.4). Therefore it is important to estimate the channel output if the transmitted signals are transmitted. The next example derives the received signal $\hat{\mathbf{Y}}$ in two different scenarios. In both scenarios, the transmitted signals are the same but different transfer channels. This will help us estimate the received signals produced by the different transfer channels.

Example 5.2.5

This example estimates the received signal $\hat{\mathbf{Y}}$ with the help of the transmitted signal $\mathbf{X}$ and the estimated transfer channel $\hat{\mathbf{W}}_c$ derived in equations (5.2.3) and (5.2.8) respectively.

$$\mathbf{X} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},\ \hat{\mathbf{W}}_c = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

From equation (5.2.12) the received signal is estimated as

$$\hat{\mathbf{Y}} = \mathbf{X}\,\hat{\mathbf{W}}_{\mathbf{c}} \tag{5.2.12}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.2.13}$$

In example (5.2.4) a different transfer channel $\hat{\mathbf{W}}_{\mathbf{c}}$ (equation 5.2.11) was estimated. The channel output $\hat{\mathbf{Y}}$ using the transfer channel in example (5.2.4) is calculated as

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \hat{\mathbf{W}}_{\mathbf{c}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then the received signals are estimated using the same expression in equation (5.2.12) as

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.2.14}$$

Hence from the results in equations 5.2.13 and 5.2.14 it can be concluded that during transmission, the condition or the characteristics of the channel has effect on the systems output or the signals received. Different channel characteristic produces different received signals.

5.2.3 Transmitted signal $\hat{\mathbf{X}}$ estimation knowing received signal and channel characteristics

In the next example, the unknown transmitted signals are estimated with help of the known received signals and the estimated channel.

Example 5.2.6

In equation 5.2.13 we estimate the transfer channel output (received signal) over the transfer channel in equation (5.2.8). Next is to estimate what was sent over the channel. This is estimated by the expression

$$\hat{\mathbf{X}} = \hat{\mathbf{Y}}\,\hat{\mathbf{W}}_{\mathbf{c}}^{-1} \tag{5.2.15}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\mathbf{W}}_{\mathbf{c}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

According to equation (5.2.15) to estimate the transmitted signal, the channel matrix has to be inverted and then multiplied by the information measured at the receiver. The estimated channel is a 2 x 2 matrix and its inverse is calculated as

$$\hat{\mathbf{W}}_{\mathbf{c}}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\hat{\mathbf{W}}_{\mathbf{c}}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\hat{\mathbf{W}}_{\mathbf{c}}^{-1} = \frac{1}{1} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence the transmitted signal is estimated as

$$\widehat{\mathbf{X}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.2.16}$$

Using the transfer channel in equation (5.2.11) and the received signal in equation (5.2.14), the transmitted signal is estimated as

$$\widehat{\mathbf{X}} = \hat{\mathbf{Y}}\ \hat{\mathbf{W}}_{\mathbf{c}}^{-1}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \hat{\mathbf{W}}_{\mathbf{c}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The transfer channel matrix has to be inverted and then multiplied by the estimated received signal. The inverse of the transfer channel is calculated as

$$\hat{\mathbf{W}}_{\mathbf{c}}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\mathbf{W}}_{\mathbf{c}}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\hat{\mathbf{W}}_{\mathbf{c}}^{-1} = \frac{1}{1} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Hence the transmitted signal is estimated as

$$\widehat{\mathbf{X}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.2.17}$$

Comparing the estimated transmitted signals obtained in equation (5.2.16) and equation (5.2.17) to the transmitted signals in equation (5.2.9), it is seen that the receiver was able to estimate the exact information that was transmitted over the different channel models. It should be noted that the receiver was able to estimate the transmitted signals base on the information it receives and the pre-calculated channel model. We now extend this process to a more complex 3 x 3 MIMO system in the next section.

5.3 Orthonormal basis $\mathbf{U}_x$ of transmitted signal (square case)

To further understand the operational principles of MIMO channel, we extend the same principles listed in subchapter 4.2 to a case of a 3 x 3 MIMO system where there are three antennas at both transmit and receive ends. Assuming three column transmitted signals and three column received signals each are observe over *K* uniformly spaced discrete time instances as

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \quad (5.3.1)$$

The channel can be estimated by first calculating the common signal space matrix (orthonormal base matrix). In the next example, the orthonormal base of the transmitted signal $\mathbf{X}$ is calculated using Gram-Schmidt procedure.

Example 5.3.1

Suppose we have K = 3 time samples per the transmitted and the received signals respectively of a 3 x 3 channel transmission system. If we assume that all the signals are real valued and that the system transmitted and received signals are observed as matrices [9,167]

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \quad (5.3.2)$$

The Gram-Schmidt QR-decomposition is applied to determine the orthonormal basis matrix $\mathbf{U_x}$ of the transmitted signal $\mathbf{X}$. The first column of the transmitted signal $\mathbf{X}$ is normalized to a unit vector to obtain the first orthonormal space $\mathbf{U}_x$ dimension. This gives

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}, \mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \qquad \mathbf{e}_1 = \frac{1}{\sqrt{59}} \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix} = \begin{pmatrix} 0.1302 \\ 0.3906 \\ 0.9113 \end{pmatrix}$$

where the length of $\mathbf{u}_1$ is

$$|\mathbf{u}_1| = \sqrt{(1)^2 + (3)^2 + (7)^2} = \sqrt{59}$$

Hence the first column of the orthonormal base $\mathbf{U_x}$ will be equal to $\mathbf{e_1}$.

The second column unit vector for the second dimension of orthonormal base $\mathbf{U_x}$ is produced from the next column of the transmitted signal $\mathbf{X}$. For this we subtract from the second column vector $\mathbf{x_2}$ its component on the first dimension. This component is a projection of $\mathbf{x_2}$ in direction $\mathbf{e_1}$ and this can be shown in equation (1.1.3) in appendix 1. The projection of $\mathbf{x_2}$ which produce vector $\mathbf{u_2}$ is given by the expression

$$\mathbf{u}_2 = \mathbf{x}_2 - (\mathbf{e}_1^{\mathrm{T}} \cdot \mathbf{x}_2)\, \mathbf{e}_1$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \left(\sqrt{59}/59 \quad 3\sqrt{59}/59 \quad 7\sqrt{59}/59\right) \cdot \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \frac{29\sqrt{59}}{59}\begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 29/59 \\ 87/59 \\ 203/59 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix}$$

Then vector $\mathbf{u}_2$ is normalized by dividing its column vector values by its length which gives $\mathbf{e}_2$. The length of $\mathbf{u}_2$ is given as

$$|\mathbf{u}_2| = \sqrt{(-147/59)^2 + (-28/59)^2 + (33/59)^2} = 2.5973$$

$$\mathbf{u}_2 = \begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix}, \ \mathbf{e}_2 = \frac{1}{2.5973}\begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix} = \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

Finally, the third column $\mathbf{e}_3$ is deduced from the third column of the transmitted signal $\mathbf{X}$ when we subtract its projections to the first and second dimensions and normalize it. Vector $\mathbf{u}_3$ is calculated from the expression

$$\mathbf{u}_3 = \mathbf{x}_3 - (\mathbf{e}_1^{T} \cdot \mathbf{x}_3)\mathbf{e}_1 - (\mathbf{e}_2^{T} \cdot \mathbf{x}_3)\mathbf{e}_2$$

$$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \sqrt{59}/59 & 3\sqrt{59}/59 & 7\sqrt{59}/59 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix} -$$

$$\begin{pmatrix} -0.9593 & -0.1827 & 0.2153 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 29/59 \\ 87/59 \\ 203/59 \end{pmatrix} - \begin{pmatrix} 1.5714 \\ 0.2993 \\ -0.3527 \end{pmatrix} = \begin{pmatrix} -0.0629 \\ 0.2261 \\ -0.0880 \end{pmatrix}$$

The vector $\mathbf{u}_3$ is normalized by first finding its absolute value or length to obtain the third column vector of the orthonormal basis.

$$|\mathbf{u}_3| = \sqrt{(-0.0629)^2 + (0.2261)^2 + (-0.0880)^2} = 0.2506$$

$$\mathbf{e}_3 = \frac{1}{0.2506}\begin{pmatrix} -0.0629 \\ 0.2261 \\ -0.0880 \end{pmatrix} = \begin{pmatrix} -0.2510 \\ 0.9022 \\ -0.3512 \end{pmatrix}$$

Hence, since $\mathbf{U}_\mathbf{x} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$, the orthonormal basis for the transmitted signal $\mathbf{X}$ is presented as

$$\mathbf{U}_\mathbf{x} = \begin{pmatrix} 0.1302 & -0.9593 & -0.2510 \\ 0.3906 & -0.1827 & 0.9022 \\ 0.9113 & 0.2153 & -0.3512 \end{pmatrix} \tag{5.3.3}$$

Figure 10 shows the graph of the respective independent column vectors ($\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3$) of the orthonormal basis $\mathbf{U}_\mathbf{x}$ calculated in equation (5.3.3).

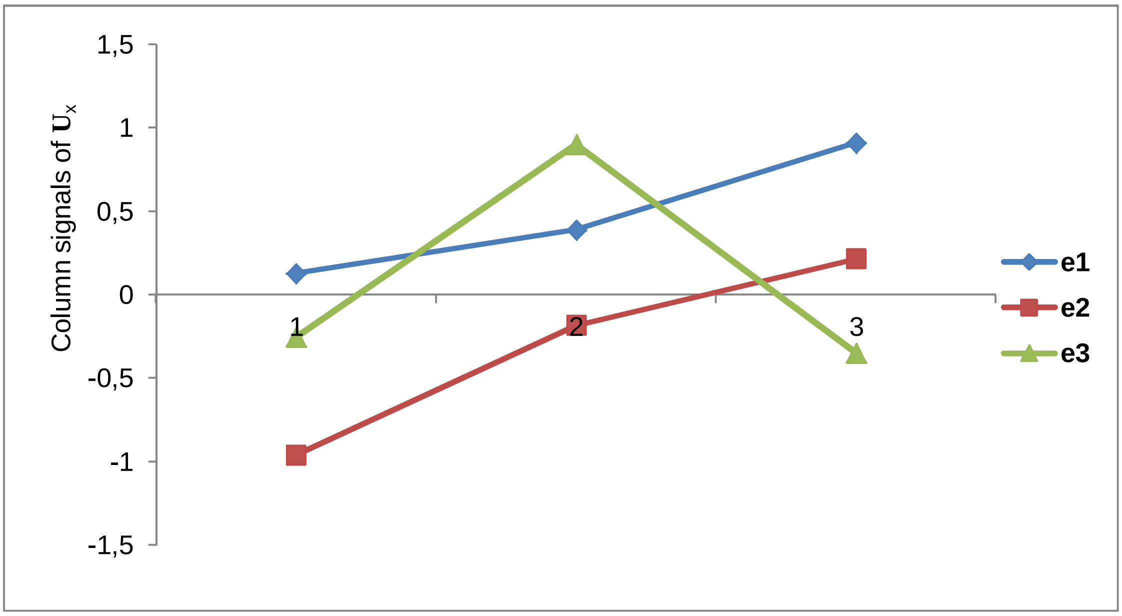

Figure 10. Column vectors of orthonormal base ($\mathbf{U}_x$).

Column vectors in equation (5.3.3) and figure 10 are orthogonal in space and therefore do not interfere with each other. The column signals of the transmitted training signal $\mathbf{X}$ of which the orthonormal basis matrix $\mathbf{U}_x$ is generated must be linearly independent signals transmitted regularly from the transmitter to the receiver.

Based on the Gram Schmidt process, the orthonormal basis $\mathbf{U}_x$ matrix is an orthogonal matrix. The column signals of the orthonormal base $\mathbf{U}_x$ in equation (5.3.3) are orthogonal and the inner product of any pair of the column vectors result zero. The column vectors are orthonormal and the norm of every column vector signal result value 1($\|\mathbf{e}_i\|=1$). To prove that the column signals of the orthonormal base $\mathbf{U}_x$ are orthogonal, the inner product of theses column signals is calculated in the next example.

Example 5.3.2

In example 5.3.1, we performed the Gram Schmidt process of the transmitted signal $\mathbf{X}$ to obtain the orthonormal basis matrix $\mathbf{U}_x$. To prove that its column vector signals are orthogonal, we calculate the inner product of any pair of the column vector signals and the result must give a value zero.

$$\mathbf{e}_1 = \begin{pmatrix} 0.1302 \\ 0.3906 \\ 0.9113 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} -0.2510 \\ 0.9022 \\ -0.3512 \end{pmatrix}$$

Taking vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, we find the inner product between these two column signals as

$\mathbf{e}_1 \cdot \mathbf{e}_2 = (0.1302\times-0.9593) + (0.3906\times-0.1827) + (0.9113\times0.2153)$

$\mathbf{e}_1 \cdot \mathbf{e}_2 = -0.00006 \approx 0$

Also the inner product between signals $\mathbf{e}_1$ and $\mathbf{e}_3$, we have

$\mathbf{e}_1 \cdot \mathbf{e}_3 = (0.1302\times-0.2510) + (0.3906\times0.9022) + (0.9113\times-0.3512)$

$\mathbf{e}_1 \cdot \mathbf{e}_3 = 0.00032 \approx 0$

The inner product between the signals $\mathbf{e}_2$ and $\mathbf{e}_3$ is also calculated as

$\mathbf{e}_2 \cdot \mathbf{e}_3 = (-0.9593\times-0.2523) + (-0.1827\times0.9026) + (0.2153\times-0.3512)$

$\mathbf{e}_2 \cdot \mathbf{e}_3 = 0.0021 \approx 0$

The results obtained from the inner product calculations prove that the column signals of the orthonormal base $\mathbf{U_x}$ are really orthogonal. It should be noted that the results obtained in example (5.3.2) are not exactly zero due to the rounding of values in calculating the orthonormal basis matrix. In the next example, we test whether the column signals of the orthonormal base matrix $\mathbf{U_x}$ obtained in example 5.3.1 are normalized by calculating the norm of each column signal, which should give a value 1.

Example 5.3.3

The norm of any column vector is calculated by summing all the squares of each vector value and finding the square root of the result. Hence the norm of the column vector signals of the orthonormal base is obtained as

$$\mathbf{U_x} = \begin{pmatrix} 0.1302 & -0.9593 & -0.2510 \\ 0.3906 & -0.1827 & 0.9022 \\ 0.9113 & 0.2153 & -0.3512 \end{pmatrix}$$

$$\|\mathbf{e}_1\| = \sqrt{(0.1302)^2 + (0.3906)^2 + (0.9113)^2} = 0.9999 \approx 1$$

$$\|\mathbf{e}_2\| = \sqrt{(-0.9593)^2 + (-0.1827)^2 + (0.2153)^2} = 0.9999 \approx 1$$

$$\|\mathbf{e}_3\| = \sqrt{(-0.2510)^2 + (0.9022)^2 + (-0.3512)^2} = 1$$

The results obtained in example (5.3.2) and (5.3.3) shows that the column vectors of the orthonormal base matrix $\mathbf{U}_x$ obtained in example (5.3.1) are normal. The number of transmitters of a transmission system may be less than the number of receivers. The next section explains how the orthonormal basis matrix $\mathbf{U_x}$ of such system can be generated.

5.4 Orthonormal basis $\mathbf{U}_x$ of transmitted signal (non square case)

Although the previous analysis involves a full rank transmitted signal $\mathbf{X}$ and receive signal $\mathbf{Y}$, it may not reflect practical situation because the number of transmitters may not be equal to the number of receivers. The decomposition of a non square transmitted signal $\mathbf{X}$ is calculated in the next example.

Example 5.4.1

If we observe K = 3 time samples per transmitted signal $\mathbf{X}$ and received signal $\mathbf{Y}$ respectively and the system now have N = 2 transmitted signals and M = 3 received signals. [9,167].

Suppose the transmitted signal $\mathbf{X}$ and received $\mathbf{Y}$ signal are given as

$$\mathbf{X} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 7 & 4 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \tag{5.4.1}$$

Referring to the Gram-Schmidt process as in appendix 1, the new orthonormal basis for the non square transmitted matrix $\mathbf{X}$ will produce the same size (3 x 3) orthonormal basis matrix $\mathbf{U}_x$ and a 2 x 3 transfer channel $\mathbf{W}_c$. Hence the orthonormal basis of the transmitted signal $\mathbf{X}$ is calculated again using Gram-Schmidt procedure. Similar to example 5.3.1, the first column of the transmitted signal $\mathbf{X}$ is normalized to a unit vector to obtain the first dimension of the orthonormal base matrix.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{e}_1 = \frac{1}{\sqrt{59}} \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix}$$

$$\mathbf{e}_1 = \begin{pmatrix} 0.1302 \\ 0.3906 \\ 0.9113 \end{pmatrix}$$

where

$$|\mathbf{u}_1| = \sqrt{(1)^2 + (3)^2 + (7)^2} = \sqrt{59}.$$

The first column of $\mathbf{U}_x$ will be equal to $\mathbf{e}_1$.

The second column unit vector for the second dimension of our orthonormal base is produced from the next column of the transmitted signal $\mathbf{X}$. We subtract from the second column vector $\mathbf{x}_2$ its component on the first dimension. That is a projection of $\mathbf{x}_2$ in direction $\mathbf{e}_1$ as shown in equation (1.1.3) in appendix 1. The vector $\mathbf{u}_2$ that is orthogonal to $\mathbf{e}_1$ is calculated from the equation

$$\mathbf{u}_2 = \mathbf{x}_2 - (\mathbf{e}_1^T \cdot \mathbf{x}_2)\, \mathbf{e}_1$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} \sqrt{59}/59 & 3\sqrt{59}/59 & 7\sqrt{59}/59 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \frac{29\sqrt{59}}{59} \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 29/59 \\ 87/59 \\ 203/59 \end{pmatrix} = \begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix}$$

Vector $\mathbf{u}_2$ is then normalized by dividing column vector values by its length which gives $\mathbf{e}_2$. The length of $\mathbf{u}_2$ is given as

$$|\mathbf{u}_2| = \sqrt{(-147/59)^2 + (-28/59)^2 + (33/59)^2} = 2.5973$$

$$\mathbf{u}_2 = \begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix}, \mathbf{e}_2 = \frac{1}{2.5973} \begin{pmatrix} -147/59 \\ -28/59 \\ 33/59 \end{pmatrix} = \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

$$\mathbf{U}_x = \begin{pmatrix} 0.1302 & -0.9593 \\ 0.3906 & -0.1827 \\ 0.9113 & 0.2153 \end{pmatrix} \tag{5.4.2}$$

Now we get two column orthonormal base vectors that are enough for a two dimensional signal space needed for two transmitted signals. However, since we have 2 transmitted column signals and 3 receiver column signals, we need a 3 dimensional common space matrix (3 dimensional orthonormal basis $\mathbf{U}_\mathbf{x}$) to map both transmitted signals and the received signals. The number of dimensions of the orthonormal basis matrix $\mathbf{U}_\mathbf{x}$ depends on the maximum number of transmitters and receivers in the MIMO system. Therefore we need to add a third column signals which is linearly independent from the first and second column of the transmitted signal $\mathbf{X}$ to generate the three dimensional orthonormal basis matrix $\mathbf{U}_\mathbf{x}$. In the next example, we generate the third column signals of the orthonormal basis matrix $\mathbf{U}_x$ by adding an additional linearly independent identity column signal.

Example 5.4.2

A randomly selected linearly independent third column signal is added to the transmitted signal. Hence the transmitted signals $\mathbf{X}$ to be decompose is of the form

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 0 \\ 7 & 4 & 1 \end{pmatrix}, \mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.3}$$

However, since the orthonormal base column of the first and second column signals of the transmitted signal $\mathbf{X}$ were generated in example (5.4.1), what is left now is to generate the third column signal $\mathbf{e}_3$ of the orthonormal base. The third column signal $\mathbf{e}_3$ of the orthonormal basis is calculated using the third column signal of the transmitted signals $\mathbf{X}$ when we subtract its projections to the first and second dimension and then normalize. The result vector $\mathbf{u_3}$ is calculated by the expression

$$\mathbf{u}_3 = \mathbf{x}_3 - (\mathbf{e}_1^{T} \cdot \mathbf{x}_3)\mathbf{e}_1 - (\mathbf{e}_2^{T} \cdot \mathbf{x}_3)\mathbf{e}_2$$

$$\mathbf{e}_1 = \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left(\sqrt{59}/59 \quad 3\sqrt{59}/59 \quad 7\sqrt{59}/59\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix} -$$

$$\left(-0.9593 \quad -0.1827 \quad 0.2153\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 7/59 \\ 21/59 \\ 49/59 \end{pmatrix} - \begin{pmatrix} -0.2065 \\ -0.0393 \\ 0.0464 \end{pmatrix} = \begin{pmatrix} 0.0879 \\ -0.3166 \\ 0.1231 \end{pmatrix}$$

The vector $\mathbf{u_3}$ is normalized by first finding its absolute value or length of

$$|\mathbf{u}_3| = \sqrt{(0.0879)^2 + (-0.3166)^2 + (0.1231)^2} = 0.3509$$

$$\mathbf{e}_3 = \frac{1}{0.3509} \begin{pmatrix} 0.0879 \\ -0.3166 \\ 0.1231 \end{pmatrix} = \begin{pmatrix} 0.2505 \\ -0.9023 \\ 0.3508 \end{pmatrix}$$

Since $\mathbf{U_x} = (\mathbf{e_1}\ \mathbf{e_2}\ \mathbf{e_3})$, the orthogonal basis matrix is given as

$$\mathbf{U_x} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2505 \\ 0.3906 & -0.1827 & -0.9023 \\ 0.9113 & 0.2153 & 0.3508 \end{pmatrix} \qquad (5.4.4)$$

Comparing equation (5.4.4) to that in equation (5.3.3) of example (5.3.1), the orthonormal base is the same even though they are generated with the help of different transmitted signal matrix. It should be noted that in vector representation, the signs of the column vector signals values do not play any important role. Figure 11 shows the column vectors of the orthonormal base in equation (5.4.4).

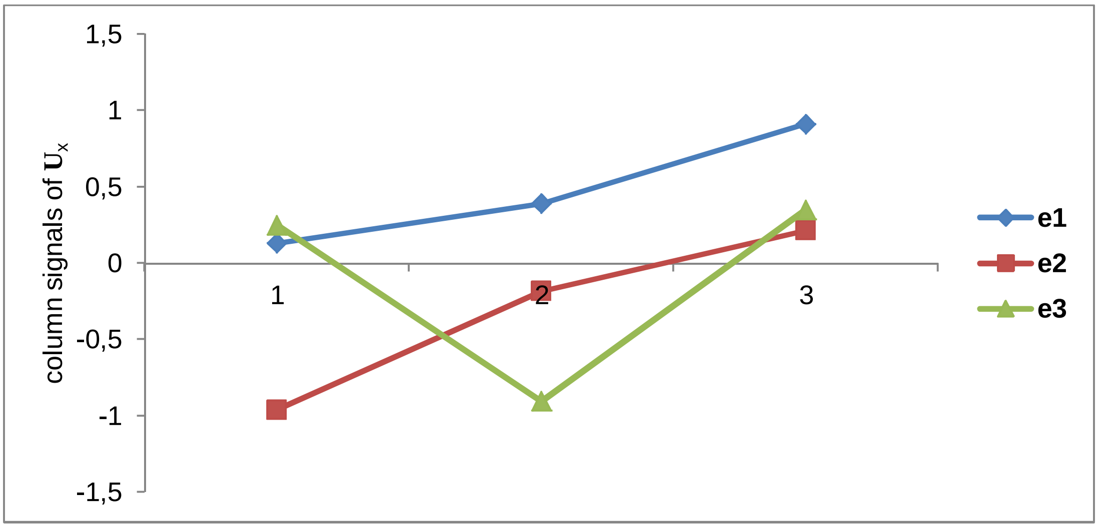

Figure 11. Column vectors of orthonormal base $\mathbf{U}_x$ (non square case).

Comparing the functions graphs of the orthonormal basis $\mathbf{U}_x$ in figure 10 and 11, it can be seen that the first and second column functions are exactly the same. However, their third column functions are the same but in opposite direction. This is because the third column signals of the transmitted signals are different in sign. The upper triangular matrix $\mathbf{R}$ to equation (5.4.4) can found in appendix 4.

To further analyze the characteristics of the orthonormal basis matrix $\mathbf{U}_x$, we modify the third identity column vector signals of the transmitted signals $\mathbf{X}$ in order to observe the orthonormal base matrix $\mathbf{U}_x$ characteristics. In the next example, we calculate only the third column vector of the orthonormal basis $\mathbf{U}_x$ matrix using the same procedure in example (5.4.2) since the first and second column signals of the transmitted signal $\mathbf{X}$ is the same as in equation (5.4.1). However, since the first and second column signals of the transmitted signals $\mathbf{X}$ is same as equation (5.4.1), their column vector values of the orthonormal basis will be the same as in equation (5.4.2) and therefore there is no need to calculate the column vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ again only $\mathbf{e}_3$ have to be calculated.

Example 5.4.3

The modified transmitted signals $\mathbf{X}$ is of the form

$$\mathbf{X}=\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 0 \\ 7 & 4 & 0 \end{pmatrix} \tag{5.4.5}$$

Also

$$\mathbf{e}_1 = \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

The third column signal $\mathbf{e}_3$ of the orthonormal base is calculated using the third column of the transmitted signals $\mathbf{X}$ when we subtract its projections to the first and second dimension and then normalized. Vector $\mathbf{u}_3$ is calculated from the expression

$$\mathbf{u}_3 = \mathbf{x}_3 - (\mathbf{e}_1^{T} \cdot \mathbf{x}_3)\mathbf{e}_1 - (\mathbf{e}_2^{T} \cdot \mathbf{x}_3)\mathbf{e}_2$$

$$\mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(\sqrt{59}/59 \quad 3\sqrt{59}/59 \quad 7\sqrt{59}/59\right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} \sqrt{59}/59 \\ 3\sqrt{59}/59 \\ 7\sqrt{59}/59 \end{pmatrix} -$$

$$\left(-0.9593 \quad -0.1827 \quad 0.2153\right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/59 \\ 3/59 \\ 7/59 \end{pmatrix} - \begin{pmatrix} 0.9203 \\ 0.1753 \\ -0.2065 \end{pmatrix} = \begin{pmatrix} 0.0628 \\ -0.2261 \\ 0.0879 \end{pmatrix}$$

The vector $\mathbf{u}_3$ is normalized by first finding its absolute value or length of $\mathbf{u}_3$.

$$|\mathbf{u}_3| = \sqrt{(0.0628)^2 + (-0.2261)^2 + (0.0879)^2} = 0.2506$$

$$\mathbf{e}_3 = \frac{1}{0.2506} \begin{pmatrix} 0.0628 \\ -0.2261 \\ 0.0879 \end{pmatrix} = \begin{pmatrix} 0.2506 \\ -0.9022 \\ 0.3508 \end{pmatrix}$$

Since $\mathbf{U_x} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$, the orthogonal basis matrix is given as

$$\mathbf{U_x} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2506 \\ 0.3906 & -0.1827 & -0.9022 \\ 0.9113 & 0.2153 & 0.3508 \end{pmatrix} \qquad (5.4.6)$$

We have calculated the orthonormal basis matrix $\mathbf{U_x}$ which gives the same column vector signals as in equation (5.3.3) and equation (5.4.4). This indicates that the last column vectors of the training signal $\mathbf{X}$ in examples example (5.4.2) and (5.4.3) can be different. Figure 12 shows the column vectors of the orthonormal basis $\mathbf{U_x}$ in equation (5.4.6).

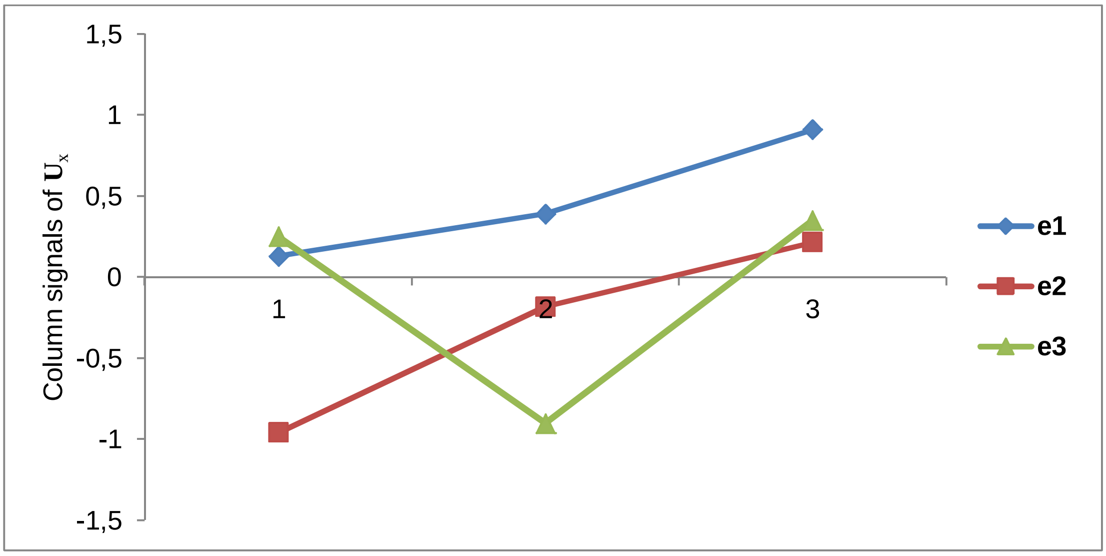

Figure 12. Orthonormal basis column functions $\mathbf{U}_x$

Comparing figure 12 to figure 11, it can be seen that both figures are the same. In MIMO systems, to determine channel characteristics the transmitted signals have to be linearly independent. The characteristics of the orthonormal base signals in the case where the transmitted signals are dependent. The next example calculates the orthonormal base using linearly dependent transmitted signal.

Example 5.4.4

In this example, we calculate the orthonormal base matrix $\mathbf{U}_x$ in the case where column signals of the transmitted signal $\mathbf{X}$ are dependent say

$$\mathbf{X} = \begin{pmatrix} -8 & -2 & -4 \\ 4 & 1 & 2 \\ 16 & 4 & 8 \end{pmatrix}, \mathbf{x}_1 = \begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} \quad (5.4.7)$$

To show that the transmitted signals are linearly dependent, we calculate such that

$$a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3 \neq 0, \text{ if a, b and c are not all zero.}$$

With the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$

$$a\begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix} + b\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + c\begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If a = 1, b = 0 and c = −2 then

$$1\begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix} + 0\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - 2\begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence the training signal $\mathbf{X}$ in equation (5.4.7) is dependent.

Now that we have proofed that the column signals of the transmitted signal in equation (5.4.7) linearly dependent, we proceed to calculate the orthonormal base vectors. The first column of the transmitted signal $\mathbf{X}$ is normalized to a unit vector to obtain the first orthonormal space $\mathbf{U}_x$ dimension. This give

$$\mathbf{u}_1 = \begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix}, \quad \mathbf{e}_1 = \frac{1}{18.3303}\begin{pmatrix} -8 \\ 4 \\ 16 \end{pmatrix} = \begin{pmatrix} -0.4364 \\ 0.2182 \\ 0.8729 \end{pmatrix}$$

Where the length of $\mathbf{u}_1$ is

$$|\mathbf{u}_1| = \sqrt{(-8)^2 + (4)^2 + (16)^2} = 4\sqrt{21} = 18.3303$$

The second column unit vector for the second dimension of orthonormal base $\mathbf{U}_x$ is produced from the next column of the transmitted signal $\mathbf{X}$. For this we subtract from the second column vector $\mathbf{x}_2$ its component on the first dimension. This component is a projection of $\mathbf{x}_2$ in direction $\mathbf{e}_1$ and this can be shown in equation (1.1.3) in appendix 2. The projection of $\mathbf{x}_2$ which produce vector $\mathbf{u}_2$ is given by the expression

$$\mathbf{u}_2 = \mathbf{x}_2 - (\mathbf{e}_1^{T} \cdot \mathbf{x}_2)\, \mathbf{e}_1$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -0.4364 & 0.2182 & 0.8729 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -0.4364 \\ 0.2182 \\ 0.8729 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - 4.5826 \begin{pmatrix} -0.4364 \\ 0.2182 \\ 0.8729 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -1.9998 \\ 0.9999 \\ 4.0001 \end{pmatrix} = \begin{pmatrix} -0.0002 \\ 0.0001 \\ 0.0001 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} -0.0002 \\ 0.0001 \\ 0.0001 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The third column $\mathbf{e}_3$ is deduced from the third column of the transmitted signal $\mathbf{X}$ when we subtract its projections to the first and second dimensions and normalize it. Vector $\mathbf{u}_3$ is calculated from the expression

$$\mathbf{u}_3 = \mathbf{x}_3 - (\mathbf{e}_1^{T} \cdot \mathbf{x}_3)\mathbf{e}_1 - (\mathbf{e}_2^{T} \cdot \mathbf{x}_3)\mathbf{e}_2$$

$$\mathbf{u}_3 = \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} -0.4364 & 0.2182 & 0.8729 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} \times \begin{pmatrix} -0.4364 \\ 0.2182 \\ 0.8729 \end{pmatrix} -$$

$$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} \times \begin{pmatrix} -0.9593 \\ -0.1827 \\ 0.2153 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} -3.9997 \\ 1.9998 \\ 8.0003 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.0003 \\ 0.0002 \\ -0.0003 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $\mathbf{U}_x = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$, the orthogonal basis for $\mathbf{X}$ is given as

$$\mathbf{U}_x = \begin{pmatrix} -0.4364 & 0 & 0 \\ 0.2182 & 0 & 0 \\ 0.8729 & 0 & 0 \end{pmatrix} \qquad (5.4.11)$$

We draw the graph of the orthogonal base matrix obtained in equation (5.4.11) in order to see it characteristics. Figure 13 shows the graph the various column vectors of the orthonormal base $\mathbf{U}_x$ (equation 5.4.11).

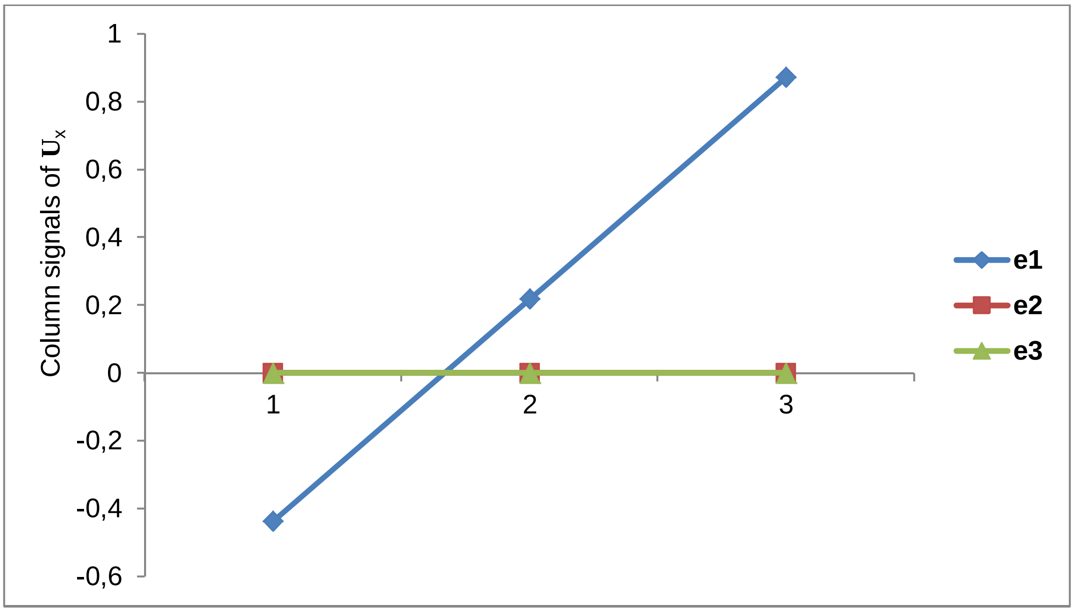

Figure 13. Column vectors of the orthonormal base $\mathbf{U}_x$.

Referring to figure 13, it can be seen that if the column signals of the transmitted signal $\mathbf{X}$ are dependent, the result of its orthonormal base produces only one dimensional space vector. This proves that we cannot model MIMO channel if the transmitted signals are dependent.

The next chapter explains how to determine the generalized Fourier coefficients of the transmitted signal $\mathbf{X}$ and received signal $\mathbf{Y}$ with the help of the orthonormal space matrix. The generalized Fourier coefficients which are coefficients of any orthogonal set of functions are needed to split both the transmitted signal $\mathbf{X}$ and received signal $\mathbf{Y}$ to help model the channel matrix.

6 Generalized Fourier coefficients of X and Y

6.1 Generalized Fourier coefficients estimation (square case)

Generalized Fourier coefficients mean coefficients of any orthogonal set of functions over which a signal is split up. Those functions are for example column vectors of the orthonormal bases $\mathbf{U_x}$ matrices in equation (5.3.3) and equation (5.4.6). The generalized Fourier coefficients of the transmitted signal $\mathbf{W}_x$ tell how much each column signal of the transmitted signal $\mathbf{X}$ contains each orthogonal column components in the orthonormal base $\mathbf{U_x}$.

Example 6.1

The transmitted signal $\mathbf{X}$ can be written as a linear combination of the mutually orthogonal column vectors $\mathbf{u}_k$, K=1,2,3, of the orthonormal basis $\mathbf{U}_x$.[9,167-168] Therefore the generalized Fourier coefficients of the transmitted signals $\mathbf{X}$ in equation (5.3.2) with respect to the orthonormal basis $\mathbf{U_x}$ in equation (5.3.3) is calculated from the expression

$$\mathbf{W}_x = \mathbf{U}_x\,\mathbf{X}$$

$$\mathbf{U_x} = \begin{pmatrix} 0.1302 & -0.9593 & -0.2510 \\ 0.3906 & -0.1827 & 0.9022 \\ 0.9113 & 0.2153 & -0.3512 \end{pmatrix},\ \mathbf{X} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}$$

$$\mathbf{W}_x=\mathbf{U}_x\mathbf{X}=\begin{pmatrix} 0.1302 & -0.9593 & -0.2510 \\ 0.3906 & -0.1827 & 0.9022 \\ 0.9113 & 0.2153 & -0.3512 \end{pmatrix}\cdot\begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix} \quad (6.1.1)$$

The generalized Fourier coefficient of $\mathbf{X}$ (equation 6.1.1) is calculated by taking the first sample of each orthonormal base column vectors and multiply them to each column samples of transmitted signal $\mathbf{X}$. For example, to calculate the sample value of - 4.5047 in equation (6.1.2), we take samples of each column vector of the orthonormal base $\mathbf{U_x}$ (0.1302 – 0.9593 -0.2510) and multiply it to the column signal $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$ of the transmitted signal $\mathbf{X}$. The same samples are multiplied to the second and third column signals of the transmitted signal $\mathbf{X}$ to obtain the values – 2.2237 and – 2.4112 of $\mathbf{W_x}$. This process is applied to all the samples in $\mathbf{U}_x$ and the column signals of $\mathbf{X}$ to determine all the values of $\mathbf{W_x}$. The values of $\mathbf{W_x}$ is given as

$$\mathbf{W}_x=\begin{pmatrix} -4.5047 & -2.2237 & -2.4112 \\ 6.1579 & 2.6449 & 3.1224 \\ -0.9012 & -3.0121 & 1.1996 \end{pmatrix} \quad (6.1.2)$$

Figure 14 shows the graph of the column vector signals of the Fourier coefficients of the transmitted signal $\mathbf{W}_x$ obtained in equation 6.2.1.

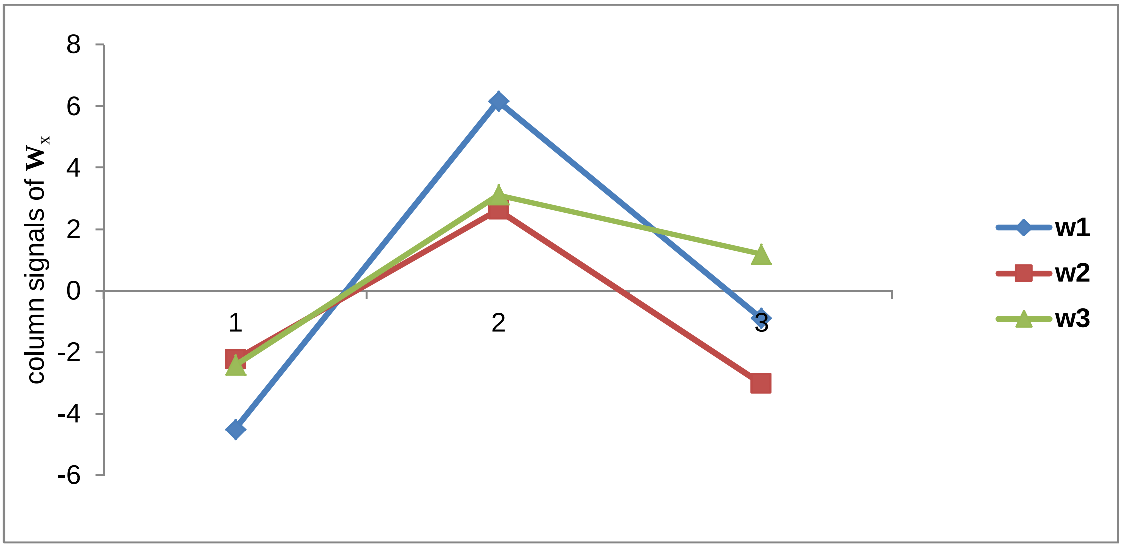

Figure 14. Function graph of $\mathbf{Wx}$ column vector signals (square case).

The column functions $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ in figure 14 indicate the coefficients of the orthogonal sets of the orthonormal base $\mathbf{U}_x$ vectors in figure 10 over which the transmitted signals in equation 5.3.2 are split up.

The generalized Fourier coefficients of the received signal $\mathbf{Y}$ with respect to the orthonormal base $\mathbf{U}_x$ are also needed to calculate channel matrix. These coefficients are determined by the projections of the column vectors of the received signal $\mathbf{Y}$ onto the orthonormal basis $\mathbf{U}_x$. In the next example, the generalized Fourier coefficients of the received signal $\mathbf{Y}$ are calculated.

Example 6.2

The Fourier coefficients of the received signal $\mathbf{Y}$ with respect to the orthonormal basis $\mathbf{U}_x$ in equation 5.2.2 is also calculated as

$$\mathbf{W}_y = \mathbf{U}_x\,\mathbf{Y}$$

$$\mathbf{U}_x = \begin{pmatrix} 0.1302 & -0.9593 & -0.2510 \\ 0.3906 & -0.1827 & 0.9022 \\ 0.9113 & 0.2153 & -0.3512 \end{pmatrix},\ \mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix}$$

$$\mathbf{W}_y = \mathbf{U}_x\,\mathbf{Y} = \begin{pmatrix} 0.1302 & -0.9593 & -0.2510 \\ 0.3906 & -0.1827 & 0.9022 \\ 0.9113 & 0.2153 & -0.3512 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \qquad (6.1.3)$$

From equation 6.1.3 we calculate the values of $\mathbf{W}_y$ using the same process applied to obtain equation 6.1.2. First we take samples of each orthonormal base $\mathbf{U}_x$ column vectors and multiply them with each column signal of the received signals. For example, to calculate the sample value of – 8.0462, we take samples of each column vector of $\mathbf{U}_x$ example (0.1302 – 0.9593 −0.2510) and multiply it with the column signal $\begin{pmatrix}1\\8\\2\end{pmatrix}$. These same samples are multiplied to the second and third column signals of the transmitted signal $\mathbf{X}$ to obtain the values – 0.2769 and 0.0541 of $\mathbf{W}_y$. This process is applied to all the samples in $\mathbf{U}_x$ and the column signals of $\mathbf{Y}$ to determine all the values of $\mathbf{W}_y$. The values of $\mathbf{W}_y$ is then given as

$$\mathbf{W}_y=\begin{pmatrix}-8.0462 & -0.2769 & 0.0541\\ 0.7334 & 8.0605 & -2.1081\\ 1.9313 & 0.9715 & 3.0915\end{pmatrix} \qquad (6.1.4)$$

Figure 15 shows the graphical characteristics of the generalized Fourier coefficients of the received signal $\mathbf{W}_y$.

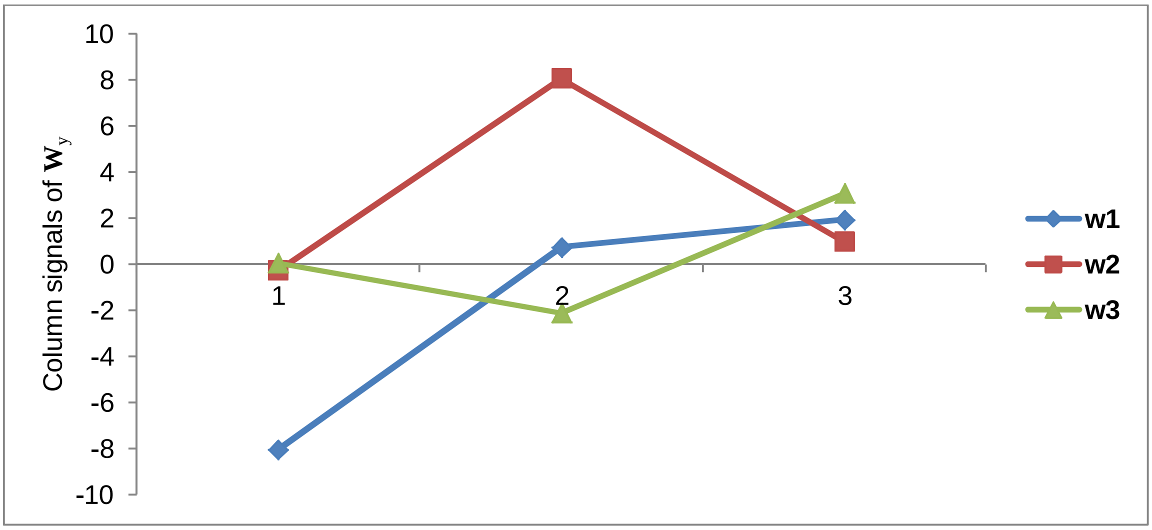

Figure 15. Function graph of $\mathbf{W}_y$ column vectors (square case).

The column functions $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ of $\mathbf{W}_y$ in figure 15 represent the first, second and third column vectors respectively. The next section explains how the generalized Fourier coefficients of the transmitted signal $\mathbf{X}$ and received signal $\mathbf{Y}$ is obtained in the case where the transmitted signal $\mathbf{X}$ is a non square matrix.

6.2 Generalized Fourier coefficients estimation (non square case)

This subchapter is to determine the generalized Fourier coefficients of both the transmitted and received vector signals using the orthonormal basis $\mathbf{U}_x$ calculated in equation (5.4.4). This is to help determine the behaviour of the transfer channel matrix. Example 6.3 explains how to find the generalized Fourier coefficients of both the transmitted signal $\mathbf{X}$ and the received signal $\mathbf{Y}$.

Example 6.3

The generalized Fourier coefficients of the transmitted signal $\mathbf{X}$ (equation (5.4.1)) with respect to the orthonormal basis $\mathbf{U_x}$ obtained in equation (5.4.4) is calculated as

$$\mathbf{U_x} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2505 \\ 0.3906 & -0.1827 & -0.9023 \\ 0.9113 & 0.2153 & 0.3508 \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 7 & 4 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix}$$

then

$$\mathbf{W_x} = \mathbf{U_x}\,\mathbf{X} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2505 \\ 0.3906 & -0.1827 & -0.9023 \\ 0.9113 & 0.2153 & 0.3508 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 7 & 4 \end{pmatrix}$$

The values in $\mathbf{W}_{\mathrm{x}}$ are calculated by taking samples of each orthonormal basis $\mathbf{U_x}$ column vectors and multiplying them with each transmitted column signals. For example we obtain the signal value – 0.9942 in $\mathbf{W_x}$ by taking sample values of each orthonormal basis $\mathbf{U_x}$ column vectors (0.1303 – 0.9593 0.2525) and multiplying them with the column signals of the transmitted signal $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$. The same samples are multiplied to the second column signals of the transmitted signal $\mathbf{X}$ to obtain the next value – 0.2177 in $\mathbf{W_x}$. The process is applied to the all the column sample values of $\mathbf{U_x}$ and column signals of $\mathbf{X}$ to obtain the generalized Fourier coefficient values of $\mathbf{W_x}$.

$$\mathbf{W_x} = \begin{pmatrix} -0.9942 & -0.2177 \\ -6.4736 & -4.5731 \\ 4.0128 & -0.2041 \end{pmatrix} \qquad (6.2.1)$$

We obtain the values in $\mathbf{W_y}$ in equation (6.2.2) similar to the process in obtaining the values in equation (6.2.1) by multiplying samples of each column vectors of $\mathbf{U_x}$ with each column signals of the received signal $\mathbf{Y}$. The generalized Fourier coefficient of the received signal $\mathbf{Y}$ is obtained from the expression

$$\mathbf{W_y} = \mathbf{U_x}\,\mathbf{Y}$$

$$\mathbf{W_y} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2505 \\ 0.3906 & -0.1827 & -0.9023 \\ 0.9113 & 0.2153 & 0.3508 \end{pmatrix} \times \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix}$$

$$\mathbf{W_y}=\begin{pmatrix} -7.0432 & 3.2336 & -1.4504 \\ -2.8756 & -4.5710 & 3.3054 \\ 3.3353 & 5.8855 & 0.9855 \end{pmatrix} \qquad (6.2.2)$$

Hence we are able to determine the generalized coefficients of both the transmitted and received signals. The next important part of this project is the estimation or the modelling of the transfer channel $\mathbf{\hat{W}_c}$. Estimating the transfer channel will help us determine how the transmitted signals are transformed to produce an output signal (received signal). In the next chapter the transfer channel is estimated in two scenarios. In the first scenario, the transfer channel is estimated with the help of the Fourier coefficients obtained in the case where all matrixes are square and the second scenario where the Fourier coefficients matrix is a non-square matrix. These Fourier coefficients are the generalized Fourier coefficients obtained in sections (6.1) and (6.2) respectively.

7 Transfer channel estimation $\mathbf{\hat{W}_c}$

7.1 Channel estimation (square case)

In the previous chapters (Chapter 5 and 6), we have successfully calculated the orthonormal basis $\mathbf{U_x}$ matrix by the decomposition of the transmitted signal $\mathbf{X}$ in the case where the transmitted signals is a square matrix and also in the case where it is not a square matrix. After obtaining the orthonormal basis matrix, we proceeded to obtain the generalized Fourier coefficients of both the transmitted signal and the received signal in both cases (square and non square) using the orthonormal basis $\mathbf{U_x}$ matrixes. In this chapter, we estimate the transfer channel matrix $\mathbf{\hat{W}_c}$ (square case) with the help of the generalized Fourier coefficients estimated in subchapter 6.1. The channel is estimated by the expression

$$\mathbf{W_y} = \mathbf{\hat{W}_c}\,\mathbf{W_x}$$

$$\mathbf{\hat{W}_c} = \mathbf{W_x}^{-1}\,\mathbf{W_y} \qquad [9,166] \qquad (7.1.1)$$

where $\mathbf{W_x}^{-1}$, is the inverse of generalized Fourier coefficients of $\mathbf{W_x}$ and $\mathbf{W_y}$ the Fourier coefficients of the received signal $\mathbf{Y}$.

However, since $\mathbf{W_x}$ is a square matrix, its inverse $\mathbf{W_x}^{-1}$ is calculated by applying the normal matrix inverse calculation process. The process of how to calculate the inverse of a 3 x 3 matrix can be found in appendix 2. Example 7.1 explains how the channel matrix is estimated.

Example 7.1

To estimate the channel transfer matrix $\hat{\mathbf{W}}_c$, the inverse of the generalized Fourier coefficient of the transmitted signal $\mathbf{W_x}$ have to be calculated and then multiply with the generalized Fourier coefficient of the received signal $\mathbf{W}_y$ as expressed in equation (7.1.1). Therefore the inverse of $\mathbf{W_x}$ in equation (6.1.2) is calculated by

$$\mathbf{W}_x = \begin{pmatrix} -4.5047 & -2.2237 & -2.4112 \\ 6.1579 & 2.6449 & 3.1224 \\ -0.9012 & -3.0121 & 1.1996 \end{pmatrix} \qquad (7.1.2)$$

First the minors of $\mathbf{W_x}$ is calculated by going through each element of the matrix ($\mathbf{W_x}$) and replacing each element by the determinant of the 2 x 2 matrix that result from deleting the elements row and column. Example we obtain the minor value 12.5778 by replacing the value – 4.5138 in equation (7.1.2) by the 2 x 2 determinant $\begin{vmatrix} 2.6449 & 3.1224 \\ -3.0121 & 1.1996 \end{vmatrix}$ which gives the minor value 12.5778 deleting the value - 4.5138 row and column. This process is performed on each value in equation (7.1.2) to obtain the minors of $\mathbf{W_x}$ as

$$\text{Minors of } \mathbf{W}_x \text{ are } \begin{pmatrix} 12.5778 & 10.2009 & -16.1646 \\ -9.9303 & -7.5768 & 11.5646 \\ -0.5659 & 0.7825 & 1.7788 \end{pmatrix} \qquad (7.1.3)$$

The cofactors of $\mathbf{W_x}$ are then determined by changing the signs of the minors. The minors are changed by applying the following signs $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

$$\text{Cofactors of } \mathbf{W}_x \text{ are } \begin{pmatrix} 12.5778 & -10.2009 & -16.1646 \\ 9.9303 & -7.5768 & -11.5646 \\ -0.5659 & -0.7825 & 1.7788 \end{pmatrix} \qquad (7.1.4)$$

The next step is to calculate the determinant of $\mathbf{W_x}$ by the sum of an element-by-element multiplication of $\mathbf{W_x}$ with the cofactors matrix. This gives the same value whichever row or column is used. Using the first row of equation (7.1.2) and the first row of the its cofactors, we obtain the determinant of $\mathbf{W_x}$ as

$\mathbf{det}(\mathbf{W_x})$=– 4.5047 (12.5778)– (–2.2237x10.2009) + (–2.4112x–16.1646)

$\mathbf{det}(\mathbf{W_x})$ = 5.0006

Next is to determine the adjoint matrix of $\mathbf{W_x}$ by finding the transpose of the cofactors of $\mathbf{W_x}$. This means that to get the adjoint of $\mathbf{W_x}$ we change each row of the cofactors to become a column

$$\mathbf{Adj}(\mathbf{W_x}) = \begin{pmatrix} 12.5778 & 9.9303 & -0.5659 \\ -10.2009 & -7.5768 & -0.7825 \\ -16.1646 & -11.5646 & 1.7788 \end{pmatrix}$$

Finally the inverse of $\mathbf{W_x}$ is found by dividing each value of the adjoint matrix by the determinant. Therefore, we obtain the inverse of $\mathbf{W_x}$ as

$$\mathbf{W_x}^{-1} = \frac{1}{5.0006}\begin{pmatrix} 12.5778 & 9.9303 & -0.5659 \\ -10.2009 & -7.5768 & -0.7825 \\ -16.1646 & -11.5646 & 1.7788 \end{pmatrix}$$

$$\mathbf{W_x}^{-1} = \begin{pmatrix} 2.5153 & 1.9858 & -0.1132 \\ -2.0399 & -1.5152 & -0.1565 \\ -3.2325 & -2.3126 & 0.3557 \end{pmatrix} \qquad (7.1.5)$$

The channel matrix is estimated by multiplying the inverse of the Fourier coefficients of the transmitted signal $\mathbf{W_x}^{-1}$ and generalized Fourier coefficients of the received signal $\mathbf{W_y}$ obtained in equation (6.1.4).

$$\hat{\mathbf{W}}_c = \mathbf{W_x}^{-1}\,\mathbf{W_y}$$

$$\hat{\mathbf{W}}_c = \begin{pmatrix} 2.5153 & 1.9858 & -0.1132 \\ -2.0399 & -1.5152 & -0.1565 \\ -3.2325 & -2.3126 & 0.3557 \end{pmatrix} \times \begin{pmatrix} -8.0462 & -0.2769 & 0.0541 \\ 0.7334 & 8.0605 & -2.1081 \\ 1.9313 & 0.9715 & 3.0915 \end{pmatrix}$$

Hence, to obtain an estimate of the transfer channel matrix $\hat{\mathbf{W}}_c$, we take samples of each column signals of $\mathbf{W_x}^{-1}$ obtained in equation (7.1.5) and multiply them with each column signals of the generalized Fourier coefficients of the received signal $\mathbf{W_y}$ to obtain each transfer channel values. Example to obtain the channel value – 19.0008 in equation (7.1.6), we take sample values of each column signals obtained in equation (7.1.5), that is values (2.5153 1.9858 – 0.1132) and multiply it with the column signal of the generalized Fourier coefficients of the received signal $\begin{pmatrix} -8.0462 \\ 0.7334 \\ 1.9313 \end{pmatrix}$. The same samples are multiplied to the second and third column signals of $\mathbf{W_y}$ to obtain the values 15.2000 and −4.4001 in equation (7.1.6). The process is applied to the all the column sample values of $\mathbf{W_x}^{-1}$ and column signals of $\mathbf{W_y}$ to estimate the channel transfer matrix as

$$\hat{\mathbf{W}}_c = \begin{pmatrix} -19.0008 & 15.2000 & -4.4001 \\ 14.9999 & -11.8005 & 2.6000 \\ 25.0002 & -17.4001 & 5.8000 \end{pmatrix} \approx \begin{pmatrix} -19 & 15.2 & -4.4 \\ 15 & -11.8 & 2.6 \\ 25 & -17.4 & 5.8 \end{pmatrix} \qquad (7.1.6)$$

Figure 16 illustrates the characteristics of the transfer channel coefficients from the transmitter to the receiver.

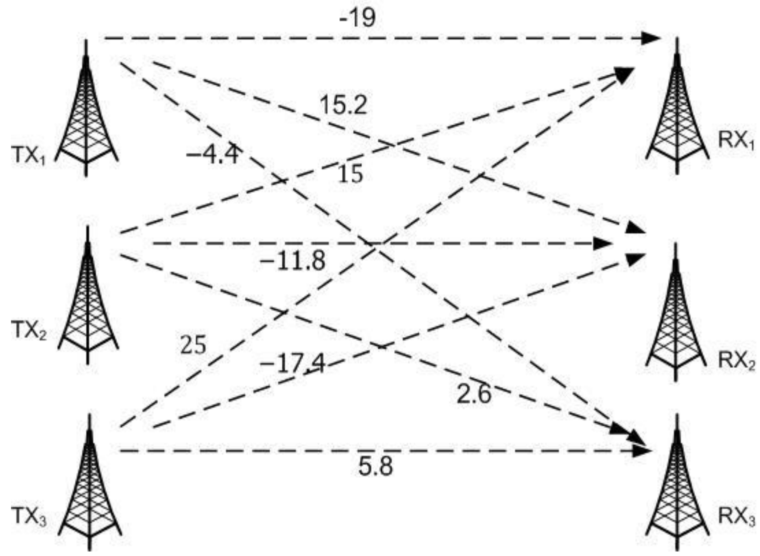

Figure 16. Estimated transfer channel $\hat{\mathbf{W}}_c$ characteristics.

In figure 16, the number −19 for example indicate the channel coefficient from TX_1 to RX_1. Also the number 15.2 indicate the channel coefficient from TX_1 to RX_2, −4.4 the channel coefficient from TX_1 to RX_3.

7.2 Transfer channel estimation (non square case)

The number of receivers may be different from the number of transmitters, $M \neq N$ and therefore it is also important to determine the characteristics. That is if we observe K = 3 time samples per $\mathbf{X}$ and per $\mathbf{Y}$ respectively and M=2 and N=3 The calculation of the transfer channel matrix $\hat{\mathbf{W}}_c$ follows the same steps as in example (7.1) by first calculating inverse of the generalized Fourier coefficient of transmitted signal $\mathbf{X}$ and the multiply it with the generalized Fourier coefficient of the received signal $\mathbf{Y}$. However, the generalized Fourier coefficients of the transmitted signal $\mathbf{W_x}$ in this case is not a square matrix and therefore the Moore-Penrose pseudo inverse of the transmitted signal $\mathbf{W_x}$ have to be calculated. Referring to equation (7.1.1) the channel transfer matrix can be estimated using the expression

$$\hat{\mathbf{W}}_c = \mathbf{W_x}^+ \mathbf{W_y}$$

where $\mathbf{W_x}^+$ is the pseudo inverse of $\mathbf{W_x}$ and $\mathbf{W_y}$ the Fourier coefficients of the received signal $\mathbf{Y}$. Mathematically, the Moore-Penrose pseudo-inverse of $\mathbf{W_x}$ can be calculated by expression

$$\mathbf{W}_x = \begin{cases} (\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1} \cdot \mathbf{W}_x^T, & M < N \\ \mathbf{W}_x^T \cdot (\mathbf{W}_x \cdot \mathbf{W}_x^T)^{-1}, & M > N \end{cases} \quad [9,165] \ (7.2.1)$$

where $\mathbf{W}_x^T$ is the transpose of Fourier coefficients of transmitted signal, M is number of transmit rows and N the number of receive columns. The next example estimates the transfer channel.

Example 7.2

In equation (7.2.1) the Moore-Penrose pseudo-inverse expresses two conditions on how the inverse the Fourier coefficients of the transmitted signals $\mathbf{W}_x$ can be calculated. The first condition satisfies this example (where N=3 and M=2) and therefore it will be used to calculate the pseudo inverse of $\mathbf{W}_x$ calculated in equation (6.2.1). The Moore-Penrose pseudo-inverse of the Fourier coefficients of the transmitted signal is calculated as

$$\mathbf{W}_x = \begin{pmatrix} -0.9942 & -0.2177 \\ -6.4736 & -4.5731 \\ 4.0128 & -0.2041 \end{pmatrix}$$

Using the first condition in equation 7.2.1

$$(\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1} \cdot \mathbf{W}_x^T, \ M < N$$

First the inverse of $(\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1}$ is calculated and multiplied by the transpose $\mathbf{W}_x^T$.

$$\mathbf{W}_x^T \cdot \mathbf{W}_x = \begin{pmatrix} -0.9942 & -6.4736 & 4.0128 \\ -0.2177 & -4.5731 & -0.2041 \end{pmatrix} \times \begin{pmatrix} -0.9942 & -0.2177 \\ -6.4736 & -4.5731 \\ 4.0128 & -0.2041 \end{pmatrix}$$

$$\mathbf{W}_x^T \cdot \mathbf{W}_x = \begin{pmatrix} 58.9985 & 29.0018 \\ 29.0018 & 21.0023 \end{pmatrix} \quad (7.2.2)$$

Since equation (7.2.2) is a 2 x 2 matrix, its inverse is calculated as the 2 x 2 inverse calculations in example (5.2.2) (equation (5.2.7)).

$$(\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1} = \begin{pmatrix} 21.0023 & -29.0018 \\ -29.0018 & 58.9985 \end{pmatrix} \times \frac{1}{(58.9985)(21.0023)-(29.0018)(29.0018)}$$

$$(\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1} = \begin{pmatrix} 0.0528 & -0.0729 \\ -0.0729 & 0.1482 \end{pmatrix}$$

$$(\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1} \cdot \mathbf{W}_x^T = \begin{pmatrix} 0.0528 & -0.0729 \\ -0.0729 & 0.1482 \end{pmatrix} \times \begin{pmatrix} -0.9942 & -6.4736 & 4.0128 \\ -0.2177 & -4.5731 & -0.2041 \end{pmatrix}$$

The Pseudo inverse of the Fourier coefficients of the transmitted signal is obtained as

$$\mathbf{W}_x^+ = (\mathbf{W}_x^T \cdot \mathbf{W}_x)^{-1} \cdot \mathbf{W}_x^T = \begin{pmatrix} -0.0366 & -0.0084 & 0.2267 \\ 0.0402 & -0.2058 & -0.3228 \end{pmatrix} \quad (7.2.3)$$

Finally, the channel matrix is estimated by taking samples of each column signals of the pseudo inverse $\mathbf{W}_x^+$ obtained in equation (7.2.3) and multiply them with each column signals of the generalized Fourier coefficients of the received signal $\mathbf{W}_y$ obtained in equation (6.2.2). For a example to obtain the channel transfer matrix value 1.0380 in equation 7.2.4 below, we take sample values of each column signals of the pseudo inverse of the transmitted signal $\mathbf{W}_x$, example (-0.0366 -0.0084 0.2267) and multiply it with the column signal of the generalized Fourier coefficients of the received signal $\begin{pmatrix} -7.0432 \\ -2.8756 \\ 3.3353 \end{pmatrix}$. The same samples are multiplied to the second and third column signals of $\mathbf{W}_y$ to obtain the values 1.2543 and 0.2487 in equation (7.2.4). The process is applied to the next column sample values of the pseudo inverse of the transmitted signal $\mathbf{W}_x^+$ and column signals of $\mathbf{W}_y$. Hence the transfer channel matrix is estimated by the expression

$$\hat{\mathbf{W}}_c = \mathbf{W}_x^+ \mathbf{W}_y$$

$$\mathbf{W}_y = \begin{pmatrix} -7.0432 & 3.2336 & -1.4504 \\ -2.8756 & -4.5710 & 3.3054 \\ 3.3353 & 5.8855 & 0.9855 \end{pmatrix}, \quad \mathbf{W}_x^+ = \begin{pmatrix} -0.0366 & -0.0084 & 0.2267 \\ 0.0402 & -0.2058 & -0.3228 \end{pmatrix}$$

$$\hat{\mathbf{W}}_c = \begin{pmatrix} -0.0366 & -0.0084 & 0.2267 \\ 0.0402 & -0.2058 & -0.3228 \end{pmatrix} \times \begin{pmatrix} -7.0432 & 3.2336 & -1.4504 \\ -2.8756 & -4.5710 & 3.3054 \\ 3.3353 & 5.8855 & 0.9855 \end{pmatrix}$$

$$\hat{\mathbf{W}}_c = \begin{pmatrix} 1.0380 & 1.2543 & 0.2487 \\ -0.7737 & -0.8291 & -1.0567 \end{pmatrix} \qquad (7.2.4)$$

Hence equation (7.2.4) indicates the channel coefficients of the 2 x 3 MIMO channel system. Estimating the received signals is also an important aspect of this project. This is to prove whether the modelled channels were able to produce accurate outputs or not. The next chapter estimates the received signals with the help of the pre calculated channels.

8 Received signal estimation $\hat{\mathbf{Y}}$, channel $\hat{\mathbf{W}}_c$, transmitted signal $\mathbf{X}$ known

8.1 Received signal (square case)

The transmission channel is pre-modelled or calculated before information is transmitted over the channel. Information transmitted is then measured at the receiver. The received signal is given as

$$\hat{\mathbf{Y}} = \mathbf{X}\,\hat{\mathbf{W}}_c \qquad (8.1.1)$$

Example 8.1

In equation (5.3.1) and equation (7.1.6), the transmitted signals and channel matrix are

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}, \ \hat{\mathbf{W}}_{\mathrm{c}} = \begin{pmatrix} -19.0008 & 15.2000 & -4.4001 \\ 14.9999 & -11.8005 & 2.6000 \\ 25.0002 & -17.4001 & 5.8000 \end{pmatrix}$$

From equation (8.1.1), the received signal is given by taking first of all transmitted column signals and multiplying them with each column of the estimated transfer channel $\hat{\mathbf{W}}_{\mathrm{c}}$. For example to obtain the received signal value 1 in equation (8.1.2), we take sample values of all column signals of the transmitted signal $\mathbf{X}$ at the first time instance, example (1 -2 2) and multiply it with the column signal of the transfer channel, example $\begin{pmatrix} -19.0008 \\ 14.9999 \\ 25.0002 \end{pmatrix}$. The same samples are multiplied to the second and third column signals of $\hat{\mathbf{W}}_{\mathrm{c}}$ to obtain the sample values 4 and 2 in equation (8.1.2). The process is applied to the second and third column sample values of the transmitted signal $\mathbf{X}$ and the column signals of the transfer channel $\hat{\mathbf{W}}_{\mathrm{c}}$. The received signal is estimated as

$$\hat{\mathbf{Y}} = \mathbf{X}\,\hat{\mathbf{W}}_{\mathrm{c}}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} -19.0008 & 15.2000 & -4.4001 \\ 14.9999 & -11.8005 & 2.6000 \\ 25.0002 & -17.4001 & 5.8000 \end{pmatrix}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 0.9998 & 4.0008 & 1.9999 \\ 7.9979 & -1.0007 & 0.9997 \\ 1.9946 & 6.9977 & -3.0007 \end{pmatrix} \approx \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \tag{8.1.2}$$

From the results obtained in equation (8.1.2) have shown that the channel modelled in equation (7.1.6) have produced a received signal identical to the received signal in equation (5.3.1). Figure 17 shows how the estimated transfer channel $\hat{\mathbf{W}}_{\mathrm{c}}$ transforms all the training signals to produce an output receive signals in equation (8.1.2).

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}, \ \hat{\mathbf{W}}_{\mathrm{c}} = \begin{pmatrix} -19 & 15.2 & -4.4 \\ 15 & -11.8 & 2.6 \\ 25 & -17.4 & 5.8 \end{pmatrix},$$

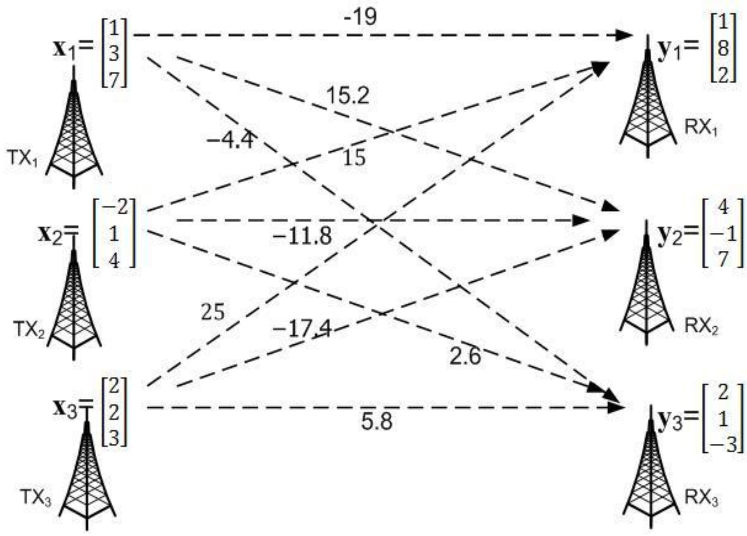

Figure 17. Channel coefficients.

In figure 17, three baseband signals $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are transmitted from the three transmitters. These signals are then transformed by the channel coefficients or characteristics and received at the receiver as $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$. The received signals are derived from the expressions

$$\hat{\mathbf{Y}} = \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3} \end{pmatrix}, \mathbf{y}_1 = \begin{pmatrix} y_{1,1} \\ y_{2,1} \\ y_{3,1} \end{pmatrix}, \mathbf{y}_2 = \begin{pmatrix} y_{1,2} \\ y_{2,2} \\ y_{3,2} \end{pmatrix}, \mathbf{y}_3 = \begin{pmatrix} y_{1,3} \\ y_{2,3} \\ y_{3,3} \end{pmatrix}$$

$$\mathbf{y}_1 = -19\mathbf{x}_1 + 15\mathbf{x}_2 + 25\mathbf{x}_3 \tag{8.1.3}$$

$$\mathbf{y}_2 = 15.2\mathbf{x}_1 + -11.8\mathbf{x}_2 + -17.4\mathbf{x}_3 \tag{8.1.4}$$

$$\mathbf{y}_3 = -4.4\mathbf{x}_1 + 2.6\mathbf{x}_2 + 5.8\mathbf{x}_3 \tag{8.1.5}$$

Equations (8.1.3), (8.1.4) and (8.1.5) show that the first received signal $\mathbf{y}_1$ is produced when each column signal of the transmitted signal $\mathbf{X}$ are weighted by the channel coefficients.

Example 8.2

In equation (8.1.3) the first column signal of the transmitted signal is weighted by the channel coefficients as

$$wx_{11} = -19\mathbf{x}_1 = -19\begin{pmatrix}1\\3\\7\end{pmatrix} = \begin{pmatrix}-19\\-57\\-133\end{pmatrix} \tag{8.1.6}$$

$$wx_{21} = 15\mathbf{x}_2 = 15\begin{pmatrix}-2\\1\\4\end{pmatrix} = \begin{pmatrix}-30\\15\\60\end{pmatrix} \tag{8.1.7}$$

$$wx_{31} = 25\mathbf{x}_3 = 25\begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}50\\50\\75\end{pmatrix} \tag{8.1.8}$$

$$\mathbf{y}_1 = \begin{pmatrix} -19 \\ -57 \\ -133 \end{pmatrix} + \begin{pmatrix} -30 \\ 15 \\ 60 \end{pmatrix} + \begin{pmatrix} 50 \\ 50 \\ 75 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix} \tag{8.1.9}$$

These weighted signals are then put together by the first receiving antenna RX_1 to produce the first received signal $\mathbf{y}_1$. Figure 18 shows the behaviour of each column transmitted signal weighted by the channel coefficients.

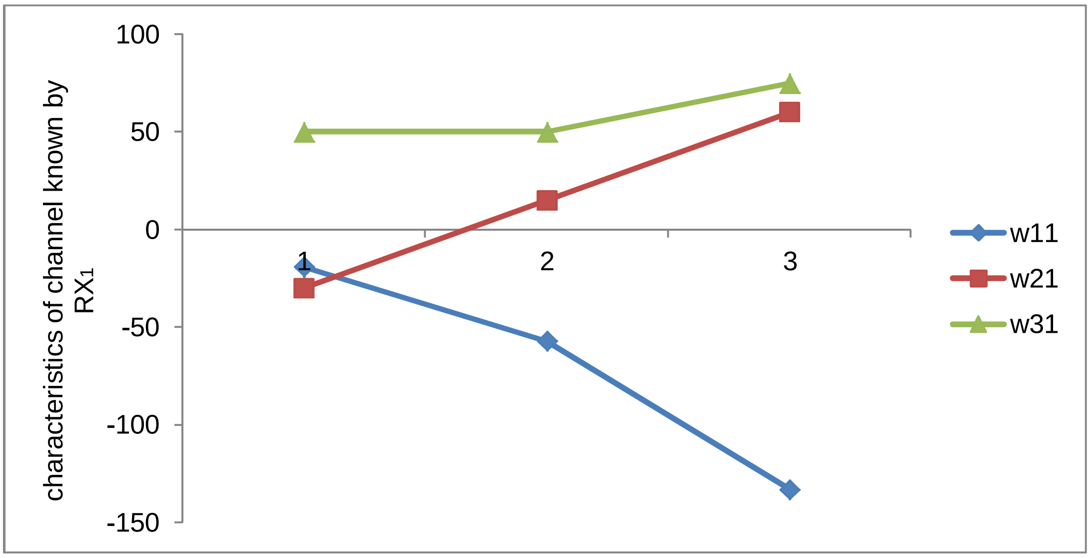

Figure 18. Weighted transmitted column signal $\mathbf{X}$ received by RX_1.

Referring to figure 18, all the weighted column signals of the transmitted signal $\mathbf{X}$ are put together to produce one received signal. Figure 19 show the graph of the first received signal.

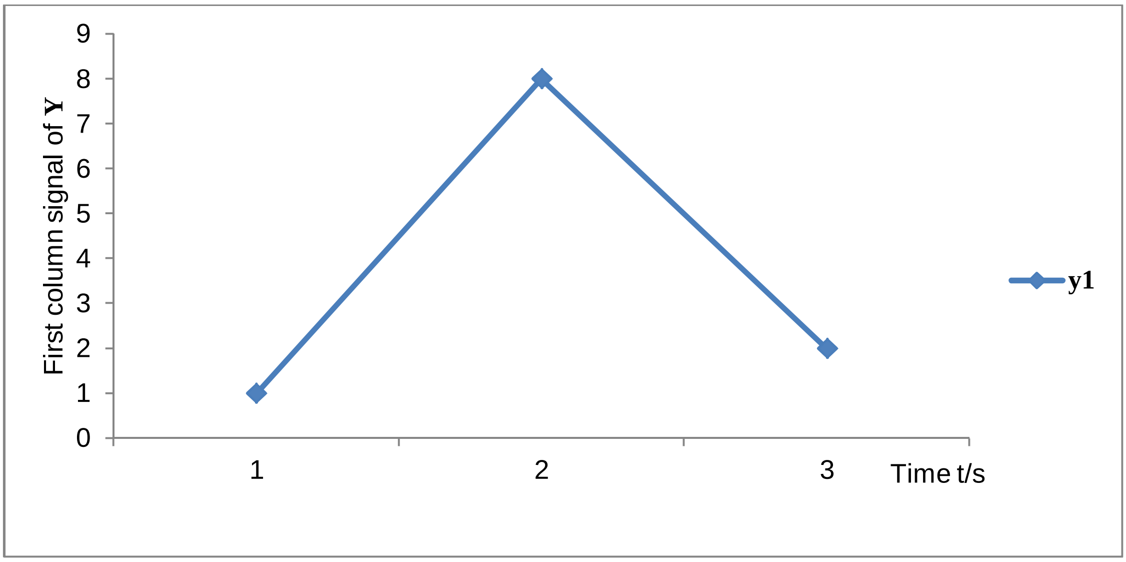

Figure 19. Function graph of first estimated received signal $\mathbf{y}_1$.

The second receiving antenna RX_2 estimated the second received signal $\mathbf{y}_2$ similar to how the first received signal was estimated but in this case with different channel coefficients.

Example 8.3

The second received signal was estimate from the expression

$$\mathbf{y}_2 = 15.2\mathbf{x}_1 + -11.8\mathbf{x}_2 + -17.4\mathbf{x}_3$$

$$wx_{12} = 15.2\mathbf{x}_1 = 15.2\begin{pmatrix}1\\3\\7\end{pmatrix} = \begin{pmatrix}15.2\\45.6\\106.4\end{pmatrix} \tag{8.1.10}$$

$$w_{22} = -11.8\mathbf{x}_2 = -11.8\begin{pmatrix}-2\\1\\4\end{pmatrix} = \begin{pmatrix}23.6\\-11.8\\-47.2\end{pmatrix} \tag{8.1.11}$$

$$w_{32} = -17.4\mathbf{x}_3 = -17.4\begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}-34.8\\-34.8\\-52.2\end{pmatrix} \tag{8.1.12}$$

$$\mathbf{y}_2 = \begin{pmatrix}15.2\\45.6\\106.4\end{pmatrix}+\begin{pmatrix}23.6\\-11.8\\-47.2\end{pmatrix}+\begin{pmatrix}-34.8\\-34.8\\-52.2\end{pmatrix} = \begin{pmatrix}4\\-1\\7\end{pmatrix} \tag{8.1.13}$$

Figure 20 also shows the behaviour of each column signal of the transmitted signals when transformed by the channel coefficients in equation (8.1.10), (8.1.11) and (8.1.12) respectively.

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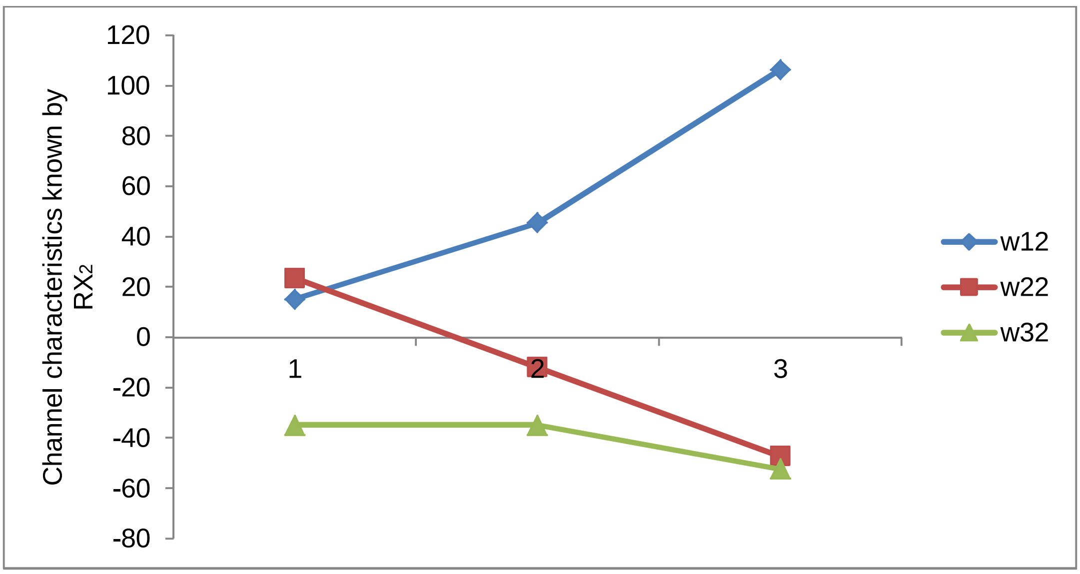

Figure 20. Weighted transmitted signals characteristics known at RX_2.

The weighted signals in figure 20 are added to produce the second received signal $\mathbf{y}_2$ as in equation (8.1.4). Figure 21 illustrate the graph of the second received signal obtained after each column signal of the transmitted signal $\mathbf{X}$ is weighted by the different channel coefficients.

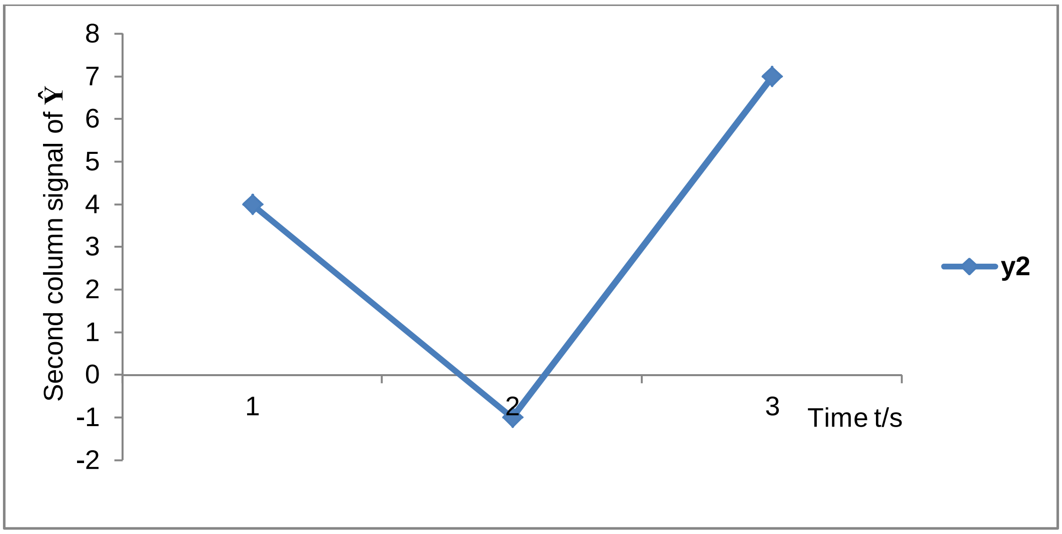

Figure 21. Function graph of second estimated received signal $\mathbf{y}_2$.

Figure 21 also shows the graph of the second received signal $\mathbf{y}_2$ when all the weighted transmitted signals are added together as in equation (8.1.13).

Finally, the third receiver RX_3 estimates the third column signal $\mathbf{y}_3$ in the same way as the first and second received signals but also with different channel coefficients.

Example 8.4

Each transmitted column signal characteristics is calculated as

$$\mathbf{y}_3 = -4.4\mathbf{x}_1 + 2.6\mathbf{x}_2 + 5.8\mathbf{x}_3$$

$$wx_{13} = -4.4\begin{pmatrix}1\\3\\7\end{pmatrix} = \begin{pmatrix}-4.4\\-13.2\\-30.8\end{pmatrix} \tag{8.1.14}$$

$$wx_{23} = 2.6\begin{pmatrix}-2\\1\\4\end{pmatrix} = \begin{pmatrix}-5.2\\2.6\\10.4\end{pmatrix} \tag{8.1.15}$$

$$wx_{33} = 5.8\begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}11.6\\11.6\\17.4\end{pmatrix} \tag{8.1.16}$$

$$\mathbf{y}_3 = \begin{pmatrix}-4.4\\-13.2\\-30.8\end{pmatrix}+\begin{pmatrix}-5.2\\2.6\\10.4\end{pmatrix}+\begin{pmatrix}11.6\\11.6\\17.4\end{pmatrix} = \begin{pmatrix}2\\1\\-3\end{pmatrix} \tag{8.1.17}$$

Figure 22 illustrates the characteristics of each column signal of the transmitted signal expressed in equation (8.1.14), (8.1.14) and (8.1.14) respectively.

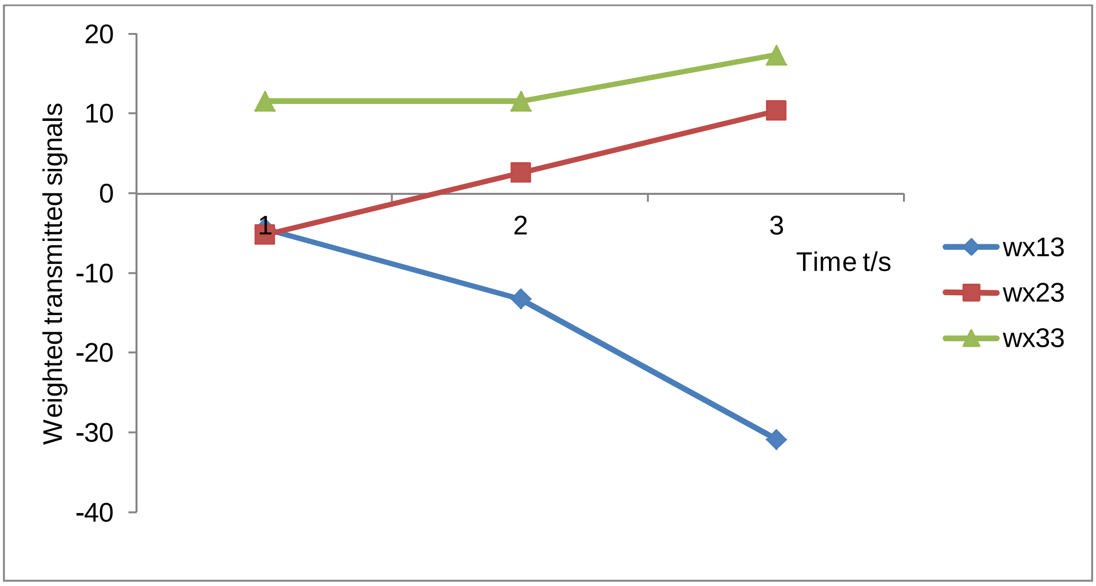

Figure 22. Weighted transmitted signals characteristics known at RX_3.

The weighted transmitted signals in figure 22 are added together to produce the third received signal $\mathbf{y}_3$ as in equation (8.1.5). Figure 23 illustrates the received signal $\mathbf{y}_3$ at RX_3.

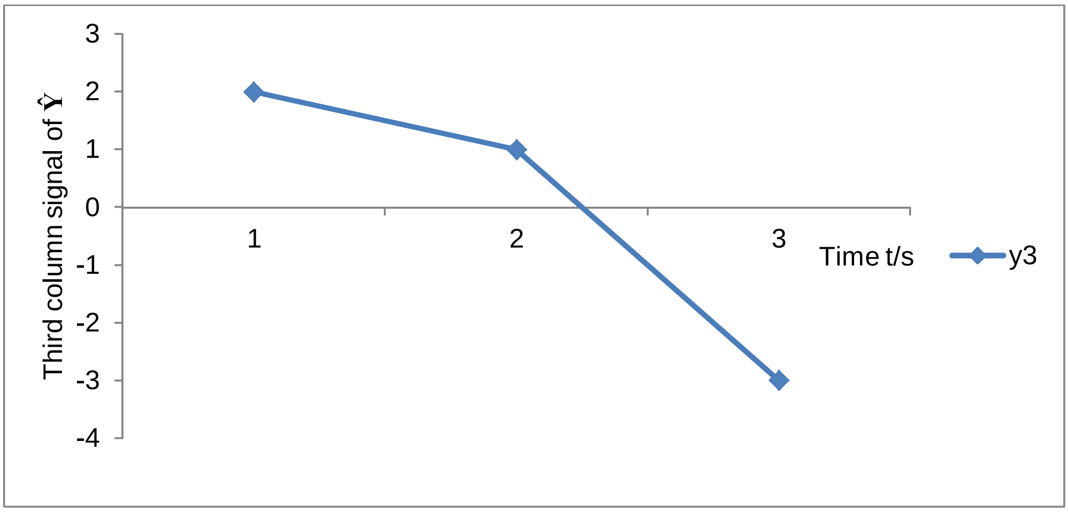

Figure 23. Function graph of third estimated received signal $\mathbf{y}_3$.

The column signals estimated in figure 19, 21 and 23 respectively by the receiving antennas are then processed together by the receiving terminal to produce one received signal $\hat{\mathbf{Y}}$ as

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3]$$

Figure 24 shows the estimated received signal $\hat{\mathbf{Y}}$ received at the receiver end of the MIMO system in figure 17.

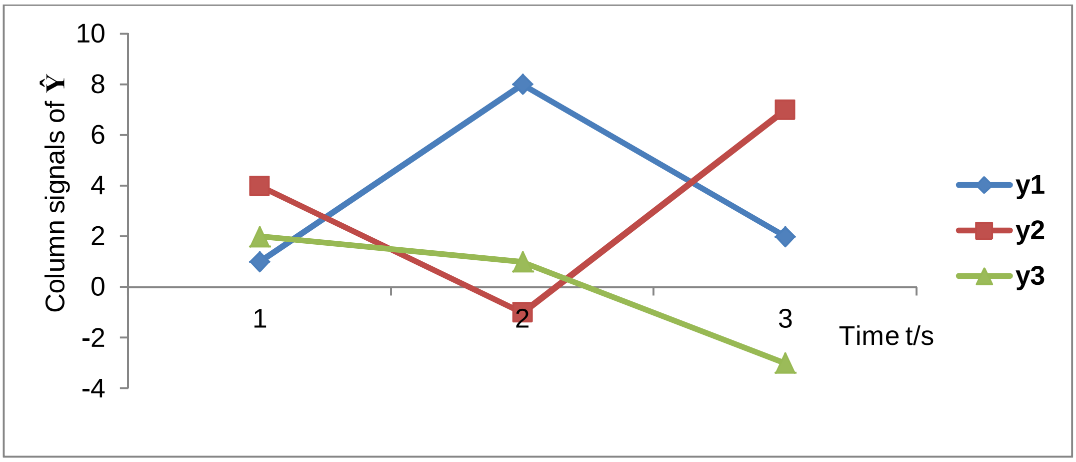

Figure 24. Function graph of estimated received signal $\hat{\mathbf{Y}}$.

Another important aspect of this project is to estimate the receive signal in the case where the transmission channel is non square. The next section estimates the received signal using the channel modelled from the 2x3 MIMO system.

8.2 Received signal (non square case)

This subchapter explains how to estimate the receive signals using the modelled channel calculated in equation (7.2.1). In subchapter (7.2), the transfer channel modelled in the 2 x 3 MIMO system was

$$\hat{\mathbf{W}}_c = \begin{pmatrix} 1.0380 & 1.2543 & 0.2487 \\ -0.7737 & -0.8291 & -1.0567 \end{pmatrix}$$

if the information transmitted (equation (5.4.2)) over the channel is

$$\mathbf{X} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 7 & 4 \end{pmatrix}.$$

Then the received signal in this case can be calculated by taking sample values of each column signals of the transmitted signal $\mathbf{X}$ and multiplying them with the column signals of the transfer channel at different time instances. The received signal can be estimated from the expression

$$\hat{\mathbf{Y}} = \mathbf{X}\,\hat{\mathbf{W}}_c$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 7 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1.0380 & 1.2543 & 0.2487 \\ -0.7737 & -0.8291 & -1.0567 \end{pmatrix}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} 2.5854 & 2.9125 & 2.3621 \\ 2.3403 & 2.9338 & -0.3106 \\ 4.1712 & 5.4637 & -2.4859 \end{pmatrix} \approx \begin{pmatrix} 2.6 & 2.9 & 2.4 \\ 2.3 & 2.9 & -0.3 \\ 4.2 & 5.5 & -2.5 \end{pmatrix} \quad (8.2.1)$$

Figure 25 shows the various transmitted signals transmitted from the two transmitting antennas and also the channel coefficients.

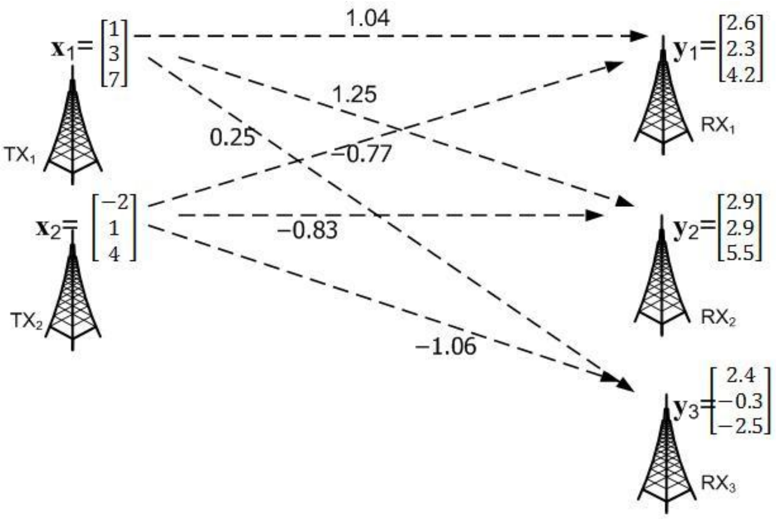

Figure 25. A 2 x 3 channel transformation of $\mathbf{X}$

From figure 25 it can be seen that the channel model does not produce the expected received signals. The estimated received signals in equation (8.2.1) and the received signals in equation (5.3.1) are not the same

$$\mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \text{ and } \hat{\mathbf{Y}} = \begin{pmatrix} 2.6 & 2.9 & 2.4 \\ 2.3 & 2.9 & -0.3 \\ 4.2 & 5.5 & -2.5 \end{pmatrix}$$

Since we do not have the exact channel transfer matrix $\hat{\mathbf{W}}_c$, the estimation quality can only be determined indirectly by finding the error norm (Frobenius norm) of the receive signal" [9,170]. The Frobenius norm is used to measure RMS (root-mean-square) gain of a matrix and its average response along given mutually orthogonal directions in space. The Frobenius norm can be calculated from the expression

$$\|\mathbf{Y} - \hat{\mathrm{Y}}\|_P = (\textstyle\sum_i |\mathbf{x_i}|^P)^{1/P} \quad [9,168]$$

where $\mathbf{x}_i$ stands for the i-th column of $\mathbf{Y} - \hat{\mathbf{Y}}$ and p = 1,2,… in this case p = 2.
First the difference between the two vectors $\mathbf{Y}$ and $\hat{\mathbf{Y}}$ is calculated as

$$\mathbf{Y} - \hat{\mathbf{Y}} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} - \begin{pmatrix} 2.5854 & 2.9125 & 2.3621 \\ 2.3403 & 2.9338 & -0.3106 \\ 4.1712 & 5.4637 & -2.4859 \end{pmatrix}$$

$$\mathbf{Y} - \hat{\mathbf{Y}} = \begin{pmatrix} -1.5854 & 1.0875 & -0.3621 \\ 5.6597 & -3.9338 & 1.3106 \\ -2.1712 & 1.5363 & -0.5141 \end{pmatrix}$$

The length of the column signals are also calculated as

$$|\mathbf{x}_1| = \sqrt{(-1.5854)^2 + (5.6597)^2 + (-2.1712)^2} = 6.2658$$

$$|\mathbf{x}_2| = \sqrt{(1.0875)^2 + (-3.9338)^2 + (1.5363)^2} = 4.3609$$

$$|\mathbf{x}_3| = \sqrt{(-0.3621)^2 + (1.3106)^2 + (-0.5141)^2} = 1.4536$$

Hence the absolute error norm is calculated as

$$\|\mathbf{Y} - \hat{\mathrm{Y}}\|_2 = \left((6.2658)^2 + (4.3609)^2 + (1.4536)^2\right)^{1/2} = 7.7711$$

The norm value explains the error between the received signal $\mathbf{Y}$ and the estimated received signal $\hat{\mathbf{Y}}$. This error can be corrected if one antenna at the receiver end of wireless link is left unused. This will change the system to a 2 x 2 MIMO system producing a 2 x 2 channel.

9 Estimation of transmitted signals

9.1 Transmitted signal estimation (square case)

Other important part in this project is the estimation of the transmitted signals over the channels that was modelled based on the known signals (known $\mathbf{X}$ and $\mathbf{Y}$). Once the receiver has some knowledge about the transmission channel, it can estimate unknown signals that have been transmitted. Therefore, there is the need to include an additional step to estimate the transmitted signals. The transmitted signal is estimated from the expression

$$\mathbf{Y} = \hat{\mathbf{W}}_c \widehat{\mathbf{X}}$$

$$\widehat{\mathbf{X}} = \mathbf{Y} \hat{\mathbf{W}}_c^{-1} \tag{9.1.1}$$

where $\hat{\mathbf{W}}_c$, the estimated transfer channel matrix obtained in equation (7.1.6) and $\mathbf{Y}$ the received signal. According to equation (9.1.1), the transmitted signal can be estimated by first finding the inverse of the estimated transfer channel matrix $\hat{\mathbf{W}}_c^{-1}$ and then multiplying it with the received signal $\mathbf{Y}$. The general process to determine the inverse of the transfer channel matrix $\hat{\mathbf{W}}_c^{-1}$ is shown in appendix 2. The estimated transmitted signal $\widehat{\mathbf{X}}$ is calculated in example 9.1.

Example 9.1

If the received signal (equation (5.3.1)) and the estimated channel matrix (equation (7.1.6)) are

$$\mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix}, \ \hat{\mathbf{W}}_c = \begin{pmatrix} -19.0008 & 15.2000 & -4.4001 \\ 14.9999 & -11.8005 & 2.6000 \\ 25.0002 & -17.4001 & 5.8000 \end{pmatrix}$$

Then from equation (9.1.1) the inverse of the channel transfer matrix $\hat{\mathbf{W}}_c$ is needed to estimate the transmitted signals. The inverse of channel transfer matrix $\hat{\mathbf{W}}_c$ is calculated by first finding the minors of $\hat{\mathbf{W}}_c$ by going through each element of the matrix and replacing each element by the determinant of the 2 x 2 matrix that result from deleting the elements row and column. Example the minor value –23.2026 is obtained by replacing the value – 19.0008 in $\hat{\mathbf{W}}_c$ by the 2 x 2 determinant $\begin{vmatrix} -11.8005 & 2.6000 \\ -17.4001 & 5.8000 \end{vmatrix}$ which gives the minor value –23.2026 deleting the value −19.0008 row and column. The process is performed on each value in $\hat{\mathbf{W}}_c$ to obtain its minors as

Minor of $\hat{\mathbf{W}}_c$ are $\begin{pmatrix} -23.2026 & 21.9989 & 34.0106 \\ 11.5978 & -0.2013 & -49.3872 \\ -12.4034 & 16.5990 & -3.7795 \end{pmatrix}$

To obtain the cofactors of $\hat{\mathbf{W}}_c$, the signs of the minors are changed by applying the following: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ and this results

Cofactors of $\hat{\mathbf{W}}_c$ are $\begin{pmatrix} -23.2026 & -21.9989 & 34.0106 \\ -11.5978 & -0.2013 & 49.3872 \\ -12.4034 & -16.5990 & -3.7795 \end{pmatrix}$

The determinant of $\hat{\mathbf{W}}_c$ can be determined by the sum of an element-by-element multiplication of $\hat{\mathbf{W}}_c$ with the cofactors matrix. This gives the same value whichever row or column is used. Hence using the first row of $\hat{\mathbf{W}}_c$ and the first row of the its cofactors, we obtain the determinant of $\hat{\mathbf{W}}_c$ as

$\mathbf{det}(\hat{\mathbf{W}}_c) = -19.0008(-23.2026) - 15.2000(21.9989) + (-4.4001)34.0106$

$\mathbf{det}(\hat{\mathbf{W}}_c) = -43.1623$

The adjoint matrix is the transpose of the cofactors of $\hat{\mathbf{W}}_c$**.** This means that to get the adjoint of $\hat{\mathbf{W}}_c$ we change each row of the cofactors to become column.

Adjoint of $\hat{\mathbf{W}}_c$ are $\begin{pmatrix} -23.2026 & -11.5978 & -12.4034 \\ -21.9989 & -0.2013 & -16.5990 \\ 34.0106 & 49.3872 & -3.7795 \end{pmatrix}$

Hence the inverse of $\hat{\mathbf{W}}_c$ by dividing each value of the adjoint matrix by the determinant. Therefore, the inverse of $\hat{\mathbf{W}}_c$ is calculated as

$$\hat{\mathbf{W}}_c^{-1} = \frac{1}{-43.1623}\begin{pmatrix} -23.2026 & -11.5978 & -12.4034 \\ -21.9989 & -0.2013 & -16.5990 \\ 34.0106 & 49.3872 & -3.7795 \end{pmatrix}$$

$$\hat{\mathbf{W}}_c^{-1} = \begin{pmatrix} 0.5376 & 0.2687 & 0.2874 \\ 0.5097 & 0.0047 & 0.3846 \\ -0.7880 & -1.1442 & 0.0876 \end{pmatrix} \qquad (9.1.2)$$

The transmitted signals are estimated by taking samples of each column signal of the received signal $\mathbf{Y}$ and multiply them with each column signals of the inverse channel transfer matrix $\hat{\mathbf{W}}_c^{-1}$. For example to obtain the transmitted signal sample value 1.0004 in equation (9.1.3) we take sample values of each column signals of the received signal $\mathbf{Y}$, example (1 4 2) and multiply it with the column signal of the inverse channel transfer matrix $\begin{pmatrix} 0.5376 \\ 0.5097 \\ -0.7880 \end{pmatrix}$. The same samples are multiplied with the second and third column signals of $\hat{\mathbf{W}}_c^{-1}$ to obtain the values – 2.0009 and 2.001 in equation (9.1.3). The process is applied to the all the column samples values of $\mathbf{Y}$ and column signals of $\hat{\mathbf{W}}_c^{-1}$.

$$\hat{\mathbf{X}} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \times \begin{pmatrix} 0.5376 & 0.2687 & 0.2874 \\ 0.5097 & 0.0047 & 0.3846 \\ -0.7880 & -1.1442 & 0.0876 \end{pmatrix}$$

The estimate of the transmitted signals is given as

$$\hat{\mathbf{X}} = \begin{pmatrix} 1.0004 & -2.0009 & 2.001 \\ 3.0031 & 1.0007 & 2.0022 \\ 7.0071 & 4.0029 & 3.0042 \end{pmatrix} \approx \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix} \qquad (9.1.3)$$

Comparing the estimated transmitted signal $\hat{\mathbf{X}}$ to the transmitted signal $\mathbf{X}$ in equation (5.3.1), the two signals are the same and we can say that MIMO works. It should be noted that the estimated transmitted signal $\hat{\mathbf{X}}$ is not exact as the transmitted signal $\mathbf{X}$ due to the approximation of decimal values in the calculation process.

9.2 Transmitted signal estimation (non square case)

In this section we estimate the transmitted signals using the non square channel model. Reference to equation (9.1.1), an estimate of the transmitted signal can be calculated using the expression

$$\hat{\mathbf{X}} = \mathbf{Y}\,\hat{\mathbf{W}}_c^{+} \qquad (9.2.1)$$

Where $\hat{\mathbf{W}}_c^{+}$, the Pseudo inverse of the transfer channel and $\mathbf{Y}$ the received signal.

If the received signal and the estimated channel are

$$\mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix},\ \hat{\mathbf{W}}_c = \begin{pmatrix} 1.0380 & 1.2543 & 0.2487 \\ -0.7737 & -0.8291 & -1.0567 \end{pmatrix}$$

This means the pseudo inverse of the transfer channel is required to estimate the transmitted signal. The pseudo inverse of the estimated channel transfer matrix is calculated using the second condition in equation (7.2.1) since the number of rows and columns of $\hat{\mathbf{W}}_c$ satisfies the condition, $M > N$, that is of receivers are higher than the number of transmitters.

$$\mathbf{W}_c^{+} = \mathbf{W}_c^{T} \cdot (\mathbf{W}_c \cdot \mathbf{W}_c^{T})^{-1},\ M > N$$

Therefore

$$\mathbf{W}_c \cdot \mathbf{W}_c^{T} = \begin{pmatrix} 1.0380 & 1.2543 & 0.2487 \\ -0.7737 & -0.8291 & -1.0567 \end{pmatrix} \times \begin{pmatrix} 1.0380 & -0.7737 \\ 1.2543 & -0.8291 \\ 0.2487 & -1.0567 \end{pmatrix}$$

$$\mathbf{W}_c \cdot \mathbf{W}_c^{T} = \begin{pmatrix} 2.7126 & -2.1058 \\ -2.1058 & 2.4026 \end{pmatrix} \qquad (9.2.2)$$

Since equation (9.2.2) is a 2 x 2 matrix, its inverse can be calculated as

$$(\mathbf{W}_c \cdot \mathbf{W}_c^{T})^{-1} = \begin{pmatrix} 2.4026 & 2.1058 \\ 2.1058 & 2.7126 \end{pmatrix} \times \frac{1}{(2.7126)(2.4026)-(2.1058)(2.1058)}$$

$$(\mathbf{W}_c \cdot \mathbf{W}_c^{T})^{-1} = \frac{1}{2.0829}\begin{pmatrix} 2.4026 & 2.1058 \\ 2.1058 & 2.7126 \end{pmatrix}$$

$$(\mathbf{W}_c \cdot \mathbf{W}_c^{T})^{-1} = \begin{pmatrix} 1.1536 & 1.0111 \\ 1.0111 & 1.3024 \end{pmatrix}$$

$$\mathbf{W}_c^{T} \cdot (\mathbf{W}_c \cdot \mathbf{W}_c^{T})^{-1} = \begin{pmatrix} 1.0380 & -0.7737 \\ 1.2543 & -0.8291 \\ 0.2487 & -1.0567 \end{pmatrix} \times \begin{pmatrix} 1.1536 & 1.0111 \\ 1.0111 & 1.3024 \end{pmatrix}$$

Hence, the Pseudo inverse of the transfer channel is calculated as

$$\mathbf{W}_c^{+} = \mathbf{W}_c^{T} \cdot (\mathbf{W}_c \cdot \mathbf{W}_c^{T})^{-1} = \begin{pmatrix} 0.4151 & 0.0419 \\ 0.6087 & 0.1884 \\ -0.7815 & -1.1248 \end{pmatrix} \qquad (9.2.3)$$

Referring to equation (9.2.1), we have to multiply the received signal $\mathbf{Y}$ with pseudo inverse of the estimated channel transfer matrix $\mathbf{W}_c^{+}$ to obtain an estimate of the transmitted signal $\hat{\mathbf{X}}$. Therefore, the estimated transmitted signal $\hat{\mathbf{X}}$ is calculated as

$$\hat{\mathbf{X}} = \mathbf{Y}\,\mathbf{W_c}^{+}$$

$$\hat{\mathbf{X}} = \begin{pmatrix} 1 & 4 & 2 \\ 8 & -1 & 1 \\ 2 & 7 & -3 \end{pmatrix} \times \begin{pmatrix} 0.4151 & 0.0419 \\ 0.6087 & 0.1884 \\ -0.7815 & -1.1248 \end{pmatrix}$$

$$\hat{\mathbf{X}} = \begin{pmatrix} 1.2869 & -1.4541 \\ 1.9306 & -0.9780 \\ 7.4356 & 4.7770 \end{pmatrix} \tag{9.2.4}$$

Comparing the estimated transmitted signal $\hat{\mathbf{X}}$ in equation (9.2.4) and the transmitted signal $\mathbf{X}$ in equation (5.4.1) it can be seen that $\hat{\mathbf{X}}$ very poorly approximate the transmitted signal $\mathbf{X}$. Since we do not have the exact channel transfer matrix $\mathbf{W_c}$, the quality or the error of the transmitted signal can only be determined by finding the error norm. [9,170]. The error norm is calculated by first finding the difference between the two vectors $\mathbf{X}$ and $\hat{\mathbf{X}}$ as

$$\mathbf{X} - \hat{\mathbf{X}} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{3} & \mathbf{1} \\ \mathbf{7} & \mathbf{4} \end{pmatrix} - \begin{pmatrix} 1.2869 & -1.4541 \\ 1.9306 & -0.9780 \\ 7.4356 & 4.7770 \end{pmatrix} = \begin{pmatrix} -0.2869 & -0.5459 \\ 1.0694 & 1.9780 \\ -0.4356 & -0.7770 \end{pmatrix}$$

Since the result will also be a vector, we find the absolute error in $\hat{\mathbf{X}}$ as

$$\left\|\mathbf{X} - \hat{\mathbf{X}}\right\|_P = \left(\sum_i \left|\mathbf{x_i}\right|^P\right)^{1/P} \qquad [9,168]$$

Where p = 1,2,…. In this case, p = 2. Therefore

$$|\mathbf{x}_1| = \sqrt{(-0.2869)^2 + (1.0694)^2 + (-0.4356)^2} = 1.1898$$

$$|\mathbf{x}_2| = \sqrt{(-0.5459)^2 + (1.9780)^2 + (-0.7770)^2} = 2.1941$$

The absolute error is calculated as

$$\left\|\mathbf{X} - \hat{\mathbf{X}}\right\|_2 = \left((1.1898)^2 + (2.1941)^2\right)^{1/2} = 2.4959$$

Hence the absolute error in the estimated transmitted signal $\hat{\mathbf{X}}$ is 2. To prevent this error, one antenna at the receiver has to be ignored or left unused so as to make the MIMO system symmetrical. This means that two antennas at both receiver and transmit end of the system will prevent the occurrence of this error.

10 Transfer channel algorithm

10.1 Introduction

In explaining the principles of how MIMO channel works, we first analyse in the chapter 5 a very simple 2 x 2 MIMO system which gives a clear understanding of the channel operations. We then extend the analysis using examples to a 3 x 3 MIMO system using examples to explain these principles. This section present a graphical output of a 10 x 10 MIMO system using an algorithm developed using Microsoft Excel® which calculates all the various steps required and the channel coefficients.

Although there are many software such as Matlab, C language, C++, etc. that can be used to develop this algorithm, they are not easy to use and not easy to acquire. The algorithm calculates all the various principles such as the orthonormal base $\mathbf{U_x}$ using Gram-Schmidt process calculates the generalized Fourier coefficients of both transmitted and received signals and the transfer channel coefficients.

10.2 A 10x10 orthonormal basis $\mathbf{U}_x$ estimation

In chapter 5 it was easy to calculate the orthonormal base because it involves 2 x 2 and 3 x 3 MIMO systems. In this chapter, an algorithm is used to calculate the orthonormal base matrix of the transmitted signal $\mathbf{X}$. Therefore if MIMO system have 10 antennas at both transmit and receive end and K = 10 time samples of the transmitted signal $\mathbf{X}$ were observed, then a 10 x 10 dimensioned orthonormal base of the transmitted signals $\mathbf{U_x}$ will be generated. For example, if we assume that the transmitted signal $\mathbf{X}$ which contains linearly independent training signals is the input values of the algorithm are

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 2 & 5 & 4 & 3 & 9 & 1 & 5 & 2 \\ 3 & 1 & 2 & 3 & 2 & 1 & 5 & 3 & 2 & 3 \\ 7 & 4 & 3 & -1 & -2 & 4 & 2 & 5 & 1 & 4 \\ 2 & 3 & 5 & 2 & -1 & 1 & 4 & 3 & 4 & 1 \\ 1 & 3 & 1 & 2 & 1 & 5 & 6 & 7 & 7 & 3 \\ 3 & 3 & 5 & 7 & 1 & 4 & 2 & 5 & 1 & 5 \\ 4 & 6 & 3 & 8 & 3 & 7 & 3 & 2 & 3 & 2 \\ 7 & 3 & 2 & 3 & 5 & 5 & 6 & 1 & 8 & 4 \\ 2 & 5 & 6 & 3 & 3 & 1 & 2 & 4 & 3 & 1 \\ 1 & 9 & 2 & 4 & 1 & 5 & 3 & 8 & 1 & 2 \end{pmatrix} \qquad (10.2.1)$$

Table 1 shows the column signal values of the generated orthonormal bases $\mathbf{U_x}$. The column of the orthonormal base are orthogonal and do not interfere with each other vectors.

Table 1. Orthonormal base $\mathbf{U}_x$ vectors for 10 dimensional signal space.

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
0.0836	-0.2614	0.3593	0.4561	0.2031	0.4125	0.2667	-0.1346	-0.5227	0.1172
0.2509	-0.1251	0.0753	0.1444	0.0320	-0.4369	0.6212	0.3863	0.0753	-0.4028
0.5854	-0.1350	-0.1721	-0.4895	-0.3610	0.2090	-0.0317	0.0725	-0.4201	-0.1145
0.1672	0.1363	0.4803	-0.2753	-0.2250	0.0368	0.3910	-0.5372	0.3617	0.1506
0.0836	0.2093	-0.0730	0.0690	0.0576	0.7482	0.1012	0.2976	0.4428	-0.2847
0.2509	0.0632	0.4221	0.3018	-0.4322	-0.0812	-0.3080	0.4654	0.1370	0.3745
0.3345	0.2725	-0.1094	0.5113	-0.1836	-0.1151	-0.2767	-0.4505	-0.0028	-0.4625
0.5854	-0.2291	-0.2691	0.0846	0.5004	-0.0659	-0.0652	-0.0960	0.3192	0.3939
0.1672	0.3245	0.5213	-0.2948	0.5474	-0.0921	-0.3139	0.1188	-0.1430	-0.2584
0.0836	0.7742	-0.2555	0.0657	0.0534	-0.0375	0.3235	0.0861	-0.2767	0.3621

Figure 26 shows the graph of the column vectors of the orthonormal basis.

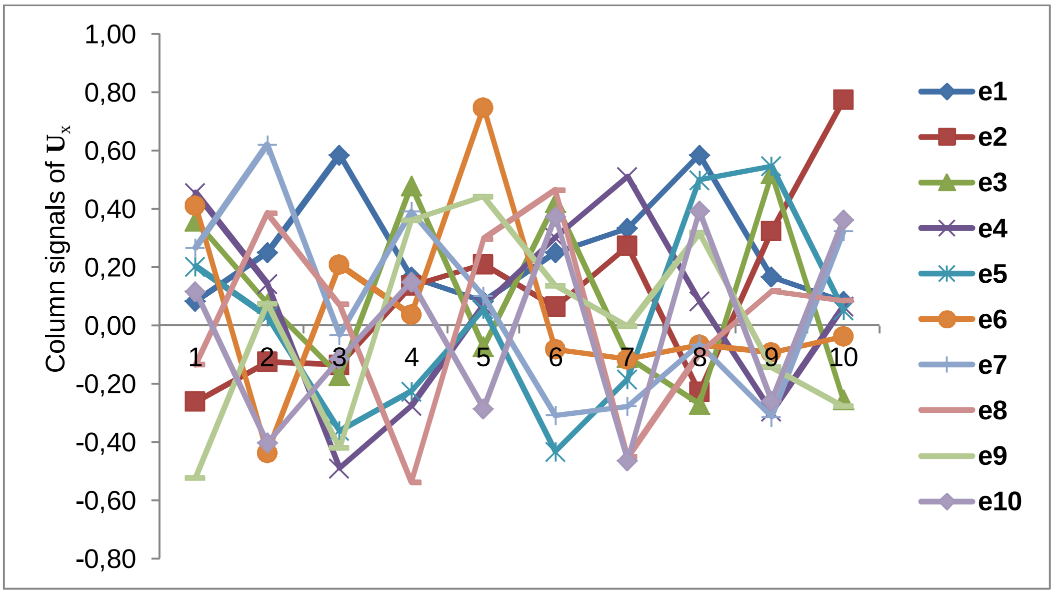

Figure 26. Graph of orthonormal base $\mathbf{U}_x$ vectors.

Referring to figure 26, if any of the column signals of the transmitted signal is changed; the pattern of the orthonormal base vectors (graph) will also change. The receiver performs this calculation (orthonormal basis $\mathbf{U}_x$) and knows the transmitted signal $\mathbf{X}$ because the transmitted signal $\mathbf{X}$ is specified initially to be linearly independent. It should be noted that if any column signal or signals of the input transmitted training signal $\mathbf{X}$ is changed, that column signal should be linearly independent to the other column signals so that the MIMO channel technique can work. The next section explains the algorithms output of the generalized Fourier coefficients of the transmitted signal $\mathbf{X}$.

10.3 Generalized Fourier coefficients of 10x10 transmitted signal

This chapter shows the graph of how the transmitted signal $\mathbf{X}$ is split using the common space matrix (orthonormal base) generated in table 1 to produce the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_x$. This calculated from the expression

$$\mathbf{W}_x = \mathbf{U}_x \mathbf{X}$$

Using the 10 x 10 algorithm, the generalized Fourier coefficients of the transmitted signal is generated by multiplying the orthonormal base values in table 1 and the transmitted signal $\mathbf{X}$ values in equation (10.2.1). Figure 27 show the graph of the generalized Fourier coefficients of the transmitted signal $\mathbf{W}_x$.

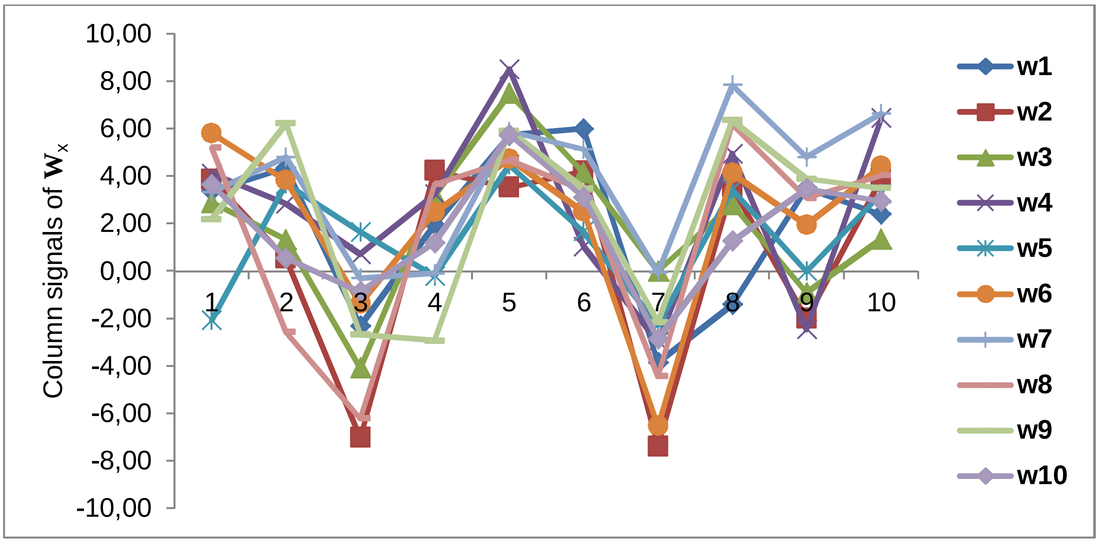

Figure 27. Function graph of Fourier coefficients of Transmitted signal $\mathbf{W}_x$.

Figure 27 tells us how much each column (signal) of the transmitted signal $\mathbf{X}$ in equation (10.2.1) contains each orthogonal column component in the orthonormal base matrix $\mathbf{U}_x$. The next section also shows how the generalized Fourier coefficients of the received signal $\mathbf{W}_y$ are generated using the algorithm.

10.4 Generalized Fourier coefficients of 10 x10 received signal

Similar to subchapter 10.3, this subchapter explains how the received signal $\mathbf{Y}$ is split with the help of the orthonormal base $\mathbf{U}_x$. The generalized Fourier coefficients of the received signal is obtain from the expression

$$\mathbf{W}_y = \mathbf{U}_x \mathbf{Y}$$

For example if the received signal at the receiver end of the 10x10 MIMO system when the transmitted training signal $\mathbf{X}$ in equation (10.2.1) is transmitted are

$$\mathbf{Y} = \begin{pmatrix} 1 & 4 & 2 & 1 & 3 & 8 & 4 & -1 & 3 & -5 \\ 8 & -1 & 1 & 2 & 5 & 6 & -2 & 3 & 3 & 3 \\ 9 & 3 & 8 & 4 & 2 & 3 & -4 & 4 & 4 & 2 \\ 2 & 7 & -3 & 5 & 1 & -6 & 2 & 2 & 1 & 7 \\ 3 & 1 & -1 & 6 & 4 & 3 & 7 & 2 & 7 & 7 \\ 3 & 2 & 2 & -4 & -1 & 1 & 1 & 5 & 7 & 2 \\ 5 & -3 & 4 & -2 & 3 & -3 & 1 & 8 & 3 & 1 \\ 4 & -1 & 3 & 2 & 4 & 4 & 4 & 2 & 2 & -2 \\ 2 & -5 & 6 & 1 & 5 & 1 & 3 & 1 & 1 & -1 \\ 1 & 3 & 7 & 3 & 6 & 1 & 8 & 3 & 2 & 1 \end{pmatrix}$$

Figure 28 shows the graphical characteristics of the generalized Fourier coefficients of received signal $\mathbf{W}_y$.

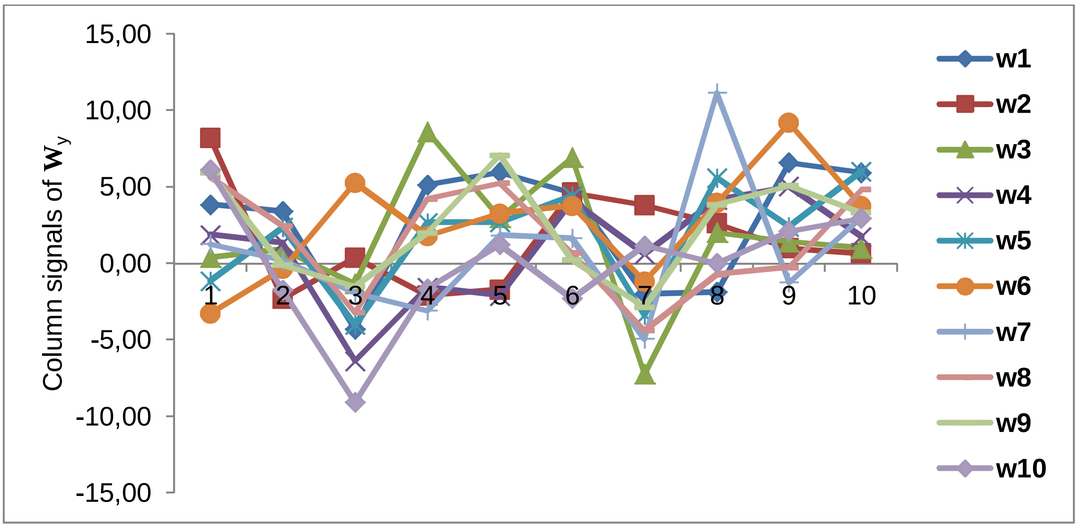

Figure 28. Function graph of generalized Fourier coefficients of received signal $\mathbf{W}_y$.

Figure 28 also tells us how much each column (signal) of the received signal $\mathbf{Y}$ contains each orthogonal column component in the orthonormal base matrix $\mathbf{U}_x$. The next section illustrates how the transfer channel matrix $\hat{\mathbf{W}}_c$ is estimated using the 10x10 Microsoft excel algorithm.

10.5 Transfer channel matrix estimation

After the calculation of $\mathbf{W}_x$ and $\mathbf{W}_y$ in subchapter 10.3 and 10.4, the next process is to estimate the transfer channel $\hat{\mathbf{W}}_c$ of the 10x10 MIMO system by first inverting the generalized Fourier coefficients of the transmitted signal and multiplying it with the column signals of the received signal. The transfer channel is obtained from the expression

$$\hat{\mathbf{W}}_c = \mathbf{W}_x^{-1}\,\mathbf{W}_y$$

Figure 29 illustrates the graph of the estimated transfer channel of the 10x10 MIMO system.

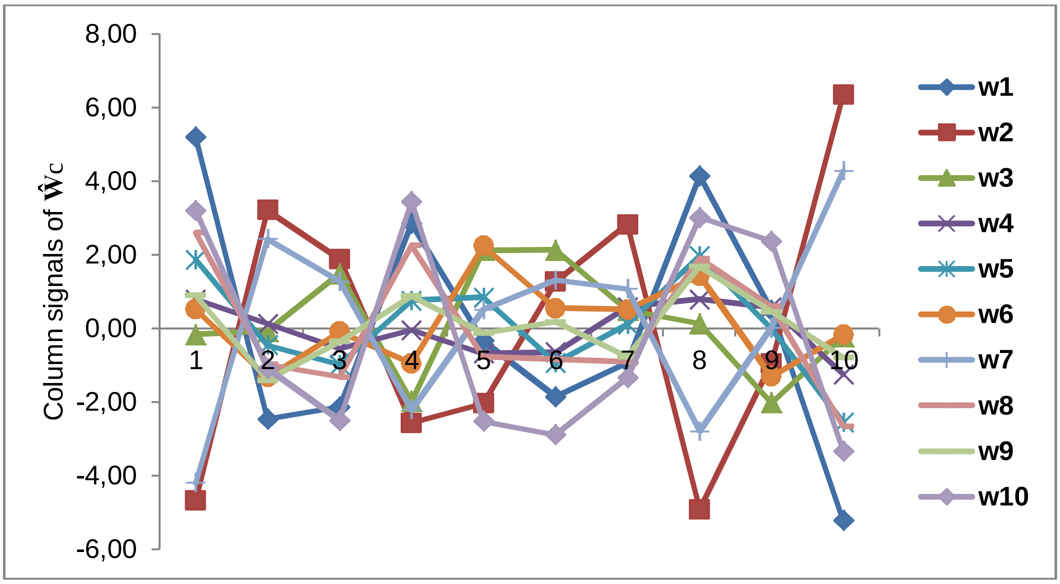

Figure 29. Function graph of the estimated transfer channel $\hat{\mathbf{W}}_c$.

The transfer channel in figure 29 is modelled with the help of known linearly independent training signals and the known received signals. If any pair of the transmitted training signal are dependent, the channel cannot be modelled. These signals are only required in the channel modelling. However, after the channel modelling dependent signals can be transmitted over the channel. Moreover, since the capacity of a MIMO system increases linearly to the number of antennas, the capacity over the 10x10 MIMO system will higher than that of the 3x3 MIMO system.

11 Results

In section (4.2) of this project, the various principles of how to achieve the goal of this project were presented. It was easy to show how a MIMO channel technology works but how it really operates is almost impossible to understand without a discrete approach. Therefore, the project was analyzed using discrete mathematical expressions and the transfer channel modelled from the analysis produced the required output.

Example (5.4.4) in section (5.4) has proved that we cannot model the MIMO channel if the known transmitted signals are dependent. This is because dependent signals produce only one-dimension orthonormal base vector. Hence linearly independent known training signals are required to model a MIMO channel to make its operation work. However, after the channel modelling unknown signals (dependent or independent signals) can be transmitted.

Furthermore, the algorithms which are easy to use were designed using Microsoft Excel® software to model the transmission channel. It also analyses all the various principles and shows their graphical characteristics. The results produced from these principles satisfied the goal of the project set in the introduction. There were situations where errors were observed but solutions on how these errors could be eliminated were provided.

12 Applications of MIMO technology

12.1 3GPP – Long Term Evolution (LTE)

One area that MIMO technology is being applied is Long Term Evolution standard for wireless communication. 3GPP stands for 3rd Generation Partnership Project. The original scope of the 3GPP was to standardise a 3rd Generation (3G) mobile system based on evolved Global Systems for Mobile (GSM) core networks and the radio access technologies that they support [11]. The 3GPP LTE is the name for release 8 of the 3GPP standard, the evolution of 3GPP Release 99. 3GPP LTE has a MIMO OFDMA (Orthogonal Frequency Division Multiple Access) physical layer on the down link and it supports various single user and multiple user MIMO modes of operation.[12].

12.2 MIMO for satellite communication

Satellite communication is one area that MIMO technology is being applied. Satellite communication systems are characterized by the strong presence of line of site link between the transmitter and the receiver. Two scenarios are considered as MIMO implementation offer potential advantage in satellite communication. The first scenario involves the use of both polarization which can result in an increase in capacity by a factor of 2 while the second scenario involves the use of a single station on earth and more than one satellite, also known as satellite diversity. The main issue in the second scenario is that the link will have unequal power since the distance from the single station on earth to the various satellites in space can vary significantly which will result in imbalance of high power. Meaning that the capacity benefit is no longer a multiplicative factor.[11]

12.3 IEEE 802.16e / WiMAX Standard

Another area that MIMO is being applied is in the IEEE 802.16 systems. The IEEE 802.16e is the mobile extension to IEEE 802.16 for wireless metropolitan area networks often known by the WiMAX (Wireless Interoperability for Microwave Access). The IEEE 802.16e has several different physical layers and MIMO modes of operation. The WiMAX mobile profile 1.0 supports OFDMA and some basic MIMO features.[12]

12.4 MIMO in the High frequency (HF) Band

The High Frequency (HF) band has been used for a number of applications including defense broadcasting, air traffic control and radio location. The use of HF radio is a cost effective way of establishing communication in regions where there is no infrastructure, for example the ocean where infrastructure is limited. Radio communication at HF most often relies on line of sight link between transmitter and the receiver since the objects in these environments are too small to act as reflectors. Measurements of MIMO HF systems have shown that the main factor limiting the ability to use multiple antenna techniques is the high correlation among links. However, using polarization or antenna pattern diversity appears to be the way to apply MIMO technology in the HF band. [11]

12.5 IEEE 802.11n channel model

The IEEE802.11n channel model was the first MIMO channel model to be accepted. This channel model covers many environments and was based on pre-existing single-input single-output (SISO) channel model developed for single wireless local area network (WLAN). The channel model is then extended to multiple antenna cases. The main feature was the concept of the cluster, that is, group of paths that have similar angles of arrival/departure as well as delays. This cluster concept was introduced by Saleh and Valenzuela in 1987. [11]

Furthermore, there are other areas that MIMO channel techniques have been applied such as MIMO for Digital Video Broadcasting (DVB-T2) that replaces the analog broadcasting, MIMO in small cellular environments, MIMO for vehicular communication for vehicular area networks, IEEE 802.11n, etc.

13 Conclusion

The goal of this project was to explain and illustrate the operation of MIMO channel technology. To achieve this goal, various principles which are difficult to understand were analyzed using a discrete mathematical approach and then extended to analyze more complex MIMO configurations. Explaining the principles of this project in the frequency domain could have produced more explanatory graphical figures but it would have been more complex to achieve the project goal. Hence, to understand the operation of MIMO, one needs to study mathematics.

The results from these principles have played a significant role in modelling the MIMO channel which is later used to prove that MIMO channel technology works. Currently MIMO technology is implemented in cellular networks at base station level, WiMAX, satellite communication systems but how can this technology be implemented in mobile devices such as smart phones since implementing multiple antennas on smart phone has become difficult. How can the MIMO channel technology improve the capacity of a wireless communication link if the distance between MIMO antennas are close to each other? These were some of the limitations which were beyond the scope of this project and are recommended for further study.

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Gram-Schmidt process

Matrix $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2,\dots,\mathbf{x}_M]$ is full column rank, i.e. all column vectors $\mathbf{x}_i$ are linearly independent. Now we factorize $\mathbf{X} = \mathbf{QR}$. First we take the first column of $\mathbf{X}$ and normalize it to a unit vector for the first dimension of our orthonormal space. This gives the first column vector $\mathbf{e}_1$ of $\mathbf{Q}$. We write

$$\mathbf{u_1} = \mathbf{x_1} \qquad \mathbf{e_1} = \frac{\mathbf{u_1}}{|\mathbf{u_1}|} \tag{1.1.1}$$

The first column of $\mathbf{Q}$ will be equal to $\mathbf{e}_1$.

The second column unit vector for the second dimension of our orthonormal space is produced from the next column of $\mathbf{X}$. For this we subtract from the second column vector $\mathbf{x}_2$ its component on the first dimension. This component is a projection of $\mathbf{x}_2$ in direction $\mathbf{e}_1$ and in general it is given by

$$\mathrm{Pr}\,oj_e\mathbf{x} = (\mathbf{e}^{\mathrm{T}} \cdot \mathbf{x})\mathbf{e} \tag{1.1.2}$$

Where $\mathbf{e}^{\mathrm{T}}$ is the transpose of $\mathbf{e}$ and $\cdot$ stands for dot or inner product (see appendix). The second column $\mathbf{e}_2$ of $\mathbf{Q}$ we get from the second column of $\mathbf{X}$ when we subtract its projection to the first orthonormal base and then normalize the result. That is:

$$\mathbf{u}_2 = \mathbf{x}_2 - (\mathbf{e}_1^{\mathrm{T}} \cdot \mathbf{x}_2)\mathbf{e}_1 \tag{1.1.3}$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{|\mathbf{u}_2|} \tag{1.1.4}$$

Figure 1.4.1 illustrates this process. As we see the resulting unit vector $\mathbf{e}_2$ will clearly be orthonormal to the first unit vector $\mathbf{e}_1$.

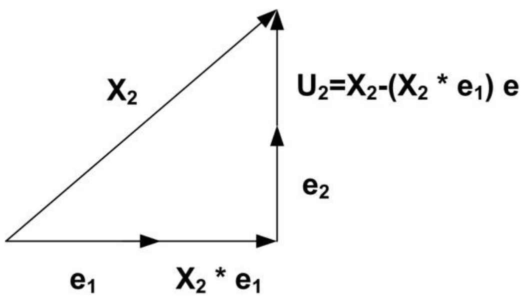

Figure 1.4.1. Two dimensional orthogonal signal space.

We can easily imagine figure 1.4.1 to be extended to third dimension, and its unit vector $\mathbf{e}_3$ is a normal to a plane of $\mathbf{e}_1$ and $\mathbf{e}_2$.

The third column $\mathbf{e_3}$ we get from the third column of $\mathbf{X}$ when we subtract from its projections to the first and second dimension and normalized it.

$$\mathbf{u_3} = \mathbf{x_3} - (\mathbf{e_1^\mathrm{T}} \cdot \mathbf{x_3})\mathbf{e_1} - (\mathbf{e}_2^\mathrm{T} \cdot \mathbf{x_3})\mathbf{e_2} \qquad \mathbf{e}_3 = \frac{\mathbf{u_3}}{|\mathbf{u_3}|}$$

We may continue to higher number of dimensions that are beyond our imagination. The fourth column we get as:

$$\mathbf{u}_4 = \mathbf{x}_4 - (\mathbf{e_1^\mathrm{T}} \cdot \mathbf{x}_4)\mathbf{e_1} - (\mathbf{e}_2^\mathrm{T} \cdot \mathbf{x}_4)\mathbf{e}_2 - (\mathbf{e}_3^\mathrm{T} \cdot \mathbf{x}_4)\mathbf{e}_3 \qquad \mathbf{e}_4 = \frac{\mathbf{u}_4}{|\mathbf{u}_4|}$$

Etc. up to M

$$\mathbf{u}_M = \mathbf{x}_M - (\mathbf{e_1^\mathrm{T}} \cdot \mathbf{x}_M)\mathbf{e_1} \cdots - (\mathbf{e}_{M-1}^\mathrm{T} \cdot \mathbf{x}_M)\mathbf{e}_{M-1} \qquad \mathbf{e}_M = \frac{\mathbf{u}_M}{|\mathbf{u}_M|}$$

This gives the orthonormal basis $\mathbf{Q}$ matrix consisting of column vectors $\mathbf{e_i}$ and it has the same dimensions K-by-M as $\mathbf{X}$.

Inverse of 3x3 matrixes

A method for finding the inverse of a 3x3 matrix is described as follows:

Using the matrix $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ as an example to illustrate this method.

1. Matrix of Minors

We go through each element of the matrix and replace it by the determinant of the 2x2 matrix that result from deleting the elements row and column.

For the example matrix, starting with the element on row 1 column 1:

$$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ \cancel{1} & 0 & 1 \\ \cancel{1} & 2 & 1 \end{pmatrix}, \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2 \text{ Gives the first element of the matrix of minors } \begin{pmatrix} -2 & & \\ & & \\ & & \end{pmatrix}$$

For the example matrix, starting with the element on row 1 and column 2:

$$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ 1 & \cancel{0} & 1 \\ 1 & \cancel{2} & 1 \end{pmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \text{ Gives the first element of the matrix of minors } \begin{pmatrix} -2 & 0 & \\ & & \\ & & \end{pmatrix}$$

In the end, we obtain the following minors: $\begin{pmatrix} -2 & 0 & 2 \\ -2 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

2. Matrix of Cofactors

In order to determine the matrix of cofactors, the signs of the matrix of minors are changed by applying the following: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

For example, the matrix of cofactors is: $\begin{pmatrix} -2 & 0 & 2 \\ 2 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}$

3. Determinant

The determinant can be found by the sum of an element-by-element multiplication of the original matrix with the cofactors matrix. It gives the same value whichever row or column is used.

For an example, choosing the top row gives determinant of $1\times\left(-2\right)+2\times 0+2\times 2=2$

Alternatively, choosing the middle column determinant of $2\times 0+0\times\left(-1\right)+2\times 1=2$

Note that if the determinant is zero then the matrix does not have an inverse. The matrix is said to be singular.

4. Adjoint

The adjoint matrix is the transpose of the matrix cofactors.

For the example, the adjoint matrix is: $\begin{pmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$.

5. Inverse

Simply the adjoint matrix multiplied by the reciprocal of the determinant, determines the inverse. For the example, the inverse is $\frac{1}{2}\begin{pmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}=\begin{pmatrix} -1 & 1 & 1 \\ 0 & -0.5 & 0.5 \\ 1 & 0 & -1 \end{pmatrix}$

Equation 5.3.3 upper triangular calculation

In reference to example 5.3.1, we calculate the values of the upper triangular matrix **R** to equation (5.3.3) by finding the inverse of the orthonormal basis matrix and then multiply it by the transmitted signal.

$$\mathbf{R} = \mathbf{U}_{\mathrm{x}}^{-1}\,\mathbf{X}$$

$$\mathbf{U_x} = \begin{pmatrix} 0.1302 & -0.9593 & -0.2523 \\ 0.3906 & -0.1827 & 0.9026 \\ 0.9113 & 0.2153 & -0.3485 \end{pmatrix}$$

$$\text{Minors of } \mathbf{Q} = \begin{pmatrix} -0.1307 & -0.9587 & 0.2506 \\ 0.3886 & 0.1845 & 0.9022 \\ -0.9120 & 0.2161 & 0.3509 \end{pmatrix}$$

$$\text{Cofactors of } \mathbf{Q} = \begin{pmatrix} -0.1307 & 0.9587 & 0.2506 \\ -0.3886 & 0.1845 & -0.9022 \\ -0.9120 & -0.2161 & 0.3509 \end{pmatrix}$$

$$\mathbf{det}(\mathbf{Q}) = 0.1302(-0.1307) - (-0.9593 \times -0.9587) + (-0.2523 \times 0.2506)$$

$$\mathbf{det}(\mathbf{Q}) = -0.9999$$

$$\mathbf{Adj}(\mathbf{Q}) = \begin{pmatrix} -0.1307 & -0.3886 & -0.9120 \\ 0.9587 & 0.1845 & -0.2161 \\ 0.2506 & -0.9022 & 0.3509 \end{pmatrix}$$

$$\mathbf{Q}^{-1} = \frac{\mathbf{Adj}(\mathbf{Q})}{\mathbf{det}(\mathbf{Q})}$$

$$\mathbf{Q}^{-1} = \frac{1}{-0.9999}\begin{pmatrix} -0.1307 & -0.3886 & -0.9120 \\ 0.9587 & 0.1845 & -0.2161 \\ 0.2506 & -0.9022 & 0.3509 \end{pmatrix}$$

$$\mathbf{Q}^{-1} = \begin{pmatrix} 0.1307 & 0.3886 & 0.9121 \\ -0.9588 & -0.1845 & 0.2161 \\ -0.2506 & 0.9023 & -0.3510 \end{pmatrix}$$

Hence **R** is calculated as

$$\mathbf{R} = \begin{pmatrix} 0.1307 & 0.3886 & 0.9121 \\ -0.9588 & -0.1845 & 0.2161 \\ -0.2506 & 0.9023 & -0.3510 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 3 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \mathbf{7}.6812 & 3.7756 & 3.7749 \\ 0.0004 & 2.5975 & -1.6383 \\ -0.0007 & -0.0005 & 0.2504 \end{pmatrix}$$

Therefore $\mathbf{R}$ can be approximate to

$$\mathbf{R} = \begin{pmatrix} \mathbf{7}.6812 & 3.7756 & 3.7749 \\ 0.0000 & 2.5975 & -1.6383 \\ 0.0000 & 0.0000 & 0.2504 \end{pmatrix}$$

Equation 5.4.4 upper triangular calculation

In reference to example 5.4.2, we calculate the upper triangular matrix $\mathbf{R}$ similar as in equation (5.4.4) by finding the inverse of the orthonormal basis matrix and then multiply it by the transmitted signal.

$$\mathbf{R} = \mathbf{U}_x^{-1}\, \mathbf{X}$$

$$\mathbf{U}_\mathbf{x} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2505 \\ 0.3906 & -0.1827 & -0.9023 \\ 0.9113 & 0.2153 & 0.3508 \end{pmatrix}$$

$$\text{Minors of } \mathbf{Q} = \begin{pmatrix} 0.1302 & 0.9593 & 0.2506 \\ -0.3905 & -0.1826 & 0.9022 \\ 0.9113 & -0.2153 & 0.3509 \end{pmatrix}$$

$$\text{Cofactors of } \mathbf{Q} = \begin{pmatrix} 0.1302 & -0.9593 & 0.2506 \\ 0.3905 & -0.1826 & -0.9022 \\ 0.9113 & 0.2153 & 0.3509 \end{pmatrix}$$

$$\textbf{Det.}\text{ of } \mathbf{Q} = 0.1302(0.1302) - (-0.9593 \times 0.9593) + 0.2505(0.2523)$$

$$\mathbf{det}(\mathbf{Q}) = 1.000$$

$$\mathbf{Adj}(\mathbf{Q}) = \begin{pmatrix} 0.1302 & 0.3905 & 0.9113 \\ -0.9593 & -0.1826 & 0.2153 \\ 0.2506 & -0.9022 & 0.3509 \end{pmatrix}$$

$$\mathbf{Q}^{-1} = \frac{\mathbf{Adj}(\mathbf{Q})}{\mathbf{det}(\mathbf{Q})}$$

$$\mathbf{Q}^{-1} = \begin{pmatrix} 0.1302 & 0.3905 & 0.9113 \\ -0.9593 & -0.1826 & 0.2153 \\ 0.2506 & -0.9022 & 0.3509 \end{pmatrix}$$

Therefore

$$\mathbf{R} = \begin{pmatrix} 0.1302 & 0.3905 & 0.9113 \\ -0.9593 & -0.1826 & 0.2153 \\ 0.2506 & -0.9022 & 0.3509 \end{pmatrix} \times \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 0 \\ 7 & 4 & 1 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 7.6812 & 3.7756 & 0.9113 \\ 0.0000 & 2.5974 & 0.2153 \\ 0.0000 & 0.0000 & 0.3509 \end{pmatrix}$$